HOMEWORK #2, ADVANCED ALGEBRA

Exercise 1. Let G be a group and a, b, c be elements of G. Prove that

(1) Elements a and $b \cdot a \cdot b^{-1}$ have equal orders;
(2) elements $a \cdot b$ and $b \cdot a$ have equal orders;
(3) elements $a \cdot b \cdot c$ and $c \cdot b \cdot a$ can have different orders.

Exercise 2. Let G be a group of order ≤ 5. Prove that G is commutative.

Exercise 3. Let G be a group. Prove that the union of two subgroups is a subgroup if and only if one subgroup is contained in the other one.

Exercise 4. Let $G = \mathbb{R} \setminus \{-1\}$. Consider the operation on G defined by the rule $x \circ y = x + y + x \cdot y$ (where $\cdot$ is the usual multiplication on real numbers). Prove that G is a group with respect to $\circ$.

Exercise 5. For which groups G the map $f : G \to G$ defined by the rule

(1) $f(x) = x^2$;
(2) $f(x) = x^{-1}$

is a homomorphism?

Exercise 6. Given a group G and two elements $x, y \in G$, one denotes $[x, y] := xyx^{-1}y^{-1}$ the *commutator* of x, y.

Which equalities below are identically true (i.e., hold for all the elements) in the symmetric group S_3?

(1) $x^6 = 1$.
(2) $[[x, y], z] = 1$.
(3) $[x^2, y^2] = 1$.

Exercise 7. Let G be a group of an even order. Prove that there exists an element $g \in G$, such that the order of g is equal to 2.

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