HOMEWORK #1, ADVANCED ALGEBRA

Exercise 1. Let $(M, \circ)$ be a monoid with the neutral element e. Assume given two elements $a, b \in M$. If $a \circ b = e$ holds then a is called a *left* inverse of b, and b is called a *right* inverse of a.

Given an element $a \in M$, assume that both left and right inverses exist for a. Prove that any left inverse of a is equal to any right inverse of a.

Exercise 2. Let $(M, \circ)$ be a semigroup. An element $e_l \in M$ (resp., $e_r \in M$) is called left neutral (resp., right neutral) if for any $a \in M$ one has $e_l \circ a = a$ (resp., $a \circ e_r = a$).

Assume that M has a left neutral element e_l. Assume further that every element a of M has a left inverse (as defined in the Exercise 1 above). Prove that

(1) The element e_l is also a right neutral (i.e., it is bilaterally neutral)
(2) Every left inverse to an element $a \in M$ is also a right inverse.
(3) Deduce from the above that M is a group.

Exercise 3. Let G be a group. Recall that the order of an element $g \in G$ is the least integer n, such that $g^n = \underbrace{g \cdot \dots \cdot g}_{n} = e$; if such an n does not exist, an element g is said to have infinite order.

Find the orders of following elements:

(1) $\frac{-\sqrt{3}}{2} + \frac{1}{2} \cdot i \in \mathbb{C}^*$ (here $\mathbb{C}^*$ is the multiplicative group of non–zero complex numbers);

(2) $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cdot i \in \mathbb{C}^*$;

(3) $\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix} \in \mathbf{GL}_2(\mathbb{C})$ (here $\mathbf{GL}_2(\mathbb{C})$ is the group of invertible complex 2×2–matrices);

(4) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in \mathbf{GL}_4(\mathbb{R})$ (here $\mathbf{GL}_4(\mathbb{C})$ is the group of invertible real 4×4–matrices)

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