APPENDIX 1. CATEGORY THEORY DEFINITIONS

1. Categories

Definition 1.1 (Category)**.** A category $\mathcal{C}$ consists of:

(1) a class $\mathrm{Ob}(\mathcal{C})$ of objects;
(2) for every ordered pair $A, B \in \mathrm{Ob}(\mathcal{C})$, a class $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of morphisms (we write $f : A \to B$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$);
(3) a composition law: for all $A, B, C \in \mathrm{Ob}(\mathcal{C})$,

$$\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C), \qquad (g, f) \longmapsto g \circ f\,;$$

(4) for each $A \in \mathrm{Ob}(\mathcal{C})$, an identity morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$.

This data should satisfy the following axioms:

(1) Associativity. For all objects $A, B, C, D \in \mathrm{Ob}(\mathcal{C})$ and all morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, $h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(2) Unit. For every morphism $f : A \to B$,

$$\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A.$$

Example 1.2. **Set**: objects are sets, morphisms are maps, id_A is the identity map.

Example 1.3. **Grp**: objects are groups, morphisms are group homomorphisms, composition and identities are the obvious ones.

Example 1.4. **Ab**: objects are abelian groups, morphisms are group homomorphisms (with the usual composition).

Example 1.5. $\mathbf{Vect}_{\mathbb{F}}$ (for a field $\mathbb{F}$): objects are vector spaces over $\mathbb{F}$, morphisms are linear maps.

2. Isomorphisms

Definition 2.1 (Isomorphism)**.** Let $\mathcal{C}$ be a category. A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ is an isomorphism if there exists a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that

$$g \circ f = \mathrm{id}_A \qquad \text{and} \qquad f \circ g = \mathrm{id}_B.$$

The morphism g is uniquely determined by f and is called the inverse of f, often denoted f^{-1}. Objects A and B are isomorphic, written $A \cong B$, if there exists an isomorphism $A \to B$.

Example 2.2. In **Set**, isomorphisms are bijective functions. In **Grp**, **Ab** and $\mathbf{Vect}_{\mathbb{F}}$, isomorphisms are bijective homomorphisms.

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3. Universal objects

Definition 3.1 (Initial object)**.** Let $\mathcal{C}$ be a category. An object $I \in \mathrm{Ob}(\mathcal{C})$ is an initial object if

$$\forall\, X \in \mathrm{Ob}(\mathcal{C}) \;\; \exists!\, f \in \mathrm{Hom}_{\mathcal{C}}(I, X).$$

Definition 3.2 (Terminal object)**.** An object $T \in \mathrm{Ob}(\mathcal{C})$ is a terminal object if

$$\forall\, X \in \mathrm{Ob}(\mathcal{C}) \;\; \exists!\, f \in \mathrm{Hom}_{\mathcal{C}}(X, T).$$

Example 3.3. In **Set**, the empty set $\emptyset$ is initial; any singleton $\{*\}$ is terminal.

Example 3.4. In **Grp**, the trivial group $\{e\}$ is both initial and terminal.

Example 3.5. In **Ab**, the trivial group $\{0\}$ is both initial and terminal.

Example 3.6. In $\mathbf{Vect}_{\mathbb{F}}$, the zero vector space $\{0\}$ is both initial and terminal.

Proposition 3.7. *If I, I' are initial objects in $\mathcal{C}$, there exists a unique isomorphism $I \cong I'$. Dually, any two terminal objects are uniquely isomorphic.*

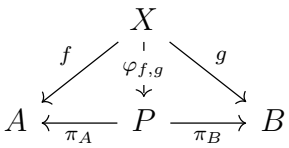
4. Products

Definition 4.1 (Product)**.** Let $\mathcal{C}$ be a category and $A, B \in \mathrm{Ob}(\mathcal{C})$. A product of A and B is a triple (P, π_A, π_B) where $P \in \mathrm{Ob}(\mathcal{C})$ and $\pi_A : P \to A$, $\pi_B : P \to B$ are morphisms satisfying the following property:

(1) For every object $X \in \mathrm{Ob}(\mathcal{C})$ and every pair of morphisms $f : X \to A$, $g : X \to B$, there exists a **unique** morphism $\varphi_{f,g} : X \to P$ such that

$$\pi_A \circ \varphi_{f,g} = f \qquad \text{and} \qquad \pi_B \circ \varphi_{f,g} = g.$$

Equivalently, the following diagram commutes:

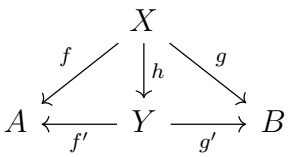


We often write $A \times B$ for the object P and call π_A, π_B the projections.

We can also give an equivalent definition.

Definition 4.2 (Product as a terminal cone)**.** Let $\mathcal{C}$ be a category and $A, B \in \mathrm{Ob}(\mathcal{C})$. The category of cones over A, B, denoted $\mathbf{Cone}(A, B)$, is defined as follows:

- objects: triples (X, f, g) with $X \in \mathrm{Ob}(\mathcal{C})$, $f : X \to A$, $g : X \to B$;
- morphisms: a morphism $h : (X, f, g) \to (Y, f', g')$ is a morphism $h : X \to Y$ in $\mathcal{C}$ such that $f' \circ h = f$ and $g' \circ h = g$, i.e. the diagram



commutes.

A product of A and B is a terminal object in $\mathbf{Cone}(A, B)$.

Example 4.3. In **Set**, the Cartesian product $A \times B$ together with the canonical projections $\pi_A(a, b) = a$, $\pi_B(a, b) = b$ is a product.

Example 4.4. In **Grp**, the direct product $G \times H$ with the natural projections is a product.

Example 4.5. In **Ab**, the direct product (or direct sum) $A \times B$ is a product.

Example 4.6. In $\mathbf{Vect}_{\mathbb{F}}$, the direct sum $V \oplus W$ together with the linear projections $(v, w) \mapsto v$ and $(v, w) \mapsto w$ is a product.

Proposition 4.7. *Let (P, π_A, π_B) and (P', π'_A, π'_B) be two products of A and B. Then there exists a unique isomorphism $\phi : P \to P'$ such that $\pi'_A \circ \phi = \pi_A$ and $\pi'_B \circ \phi = \pi_B$.*

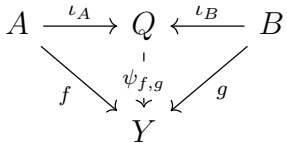
5. Coproducts

Definition 5.1 (Coproduct)**.** Let $A, B \in \mathrm{Ob}(\mathcal{C})$. A coproduct of A and B is a triple (Q, ι_A, ι_B) where $Q \in \mathrm{Ob}(\mathcal{C})$ and $\iota_A : A \to Q$, $\iota_B : B \to Q$ are morphisms satisfying the following property:

(1) For every object $Y \in \mathrm{Ob}(\mathcal{C})$ and every pair of morphisms $f : A \to Y$, $g : B \to Y$, there exists a **unique** morphism $\psi_{f,g} : Q \to Y$ such that

$$\psi_{f,g} \circ \iota_A = f \qquad \text{and} \qquad \psi_{f,g} \circ \iota_B = g.$$

In diagram form,



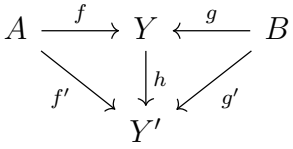
commutes.

We write $A \amalg B$ for Q and call ι_A, ι_B the coproduct injections.

We can also give an equivalent definition.

Definition 5.2 (Coproduct as an initial cocone)**.** Let $\mathcal{C}$ be a category and $A, B \in \mathrm{Ob}(\mathcal{C})$. The category of cocones under A, B, denoted $\mathbf{Cocone}(A, B)$, is defined as follows:

- objects: triples (Y, f, g) with $Y \in \mathrm{Ob}(\mathcal{C})$, $f : A \to Y$, $g : B \to Y$;
- morphisms: a morphism $h : (Y, f, g) \to (Y', f', g')$ is a morphism $h : Y \to Y'$ in $\mathcal{C}$ such that $h \circ f = f'$ and $h \circ g = g'$, i.e. the diagram



commutes.

A coproduct of A and B is an initial object in $\mathbf{Cocone}(A, B)$.

Example 5.3. In $\mathbf{Set}$, the disjoint union $A \amalg B = (A \times \{0\}) \cup (B \times \{1\})$ with the canonical injections is a coproduct.

Example 5.4. In $\mathbf{Ab}$, the direct product (or direct sum) $A \times B$ with injections $a \mapsto (a, 0)$ and $b \mapsto (0, b)$ is a coproduct.

Example 5.5. In $\mathbf{Vect}_{\mathbb{F}}$, the direct sum $V \oplus W$ with the injections $v \mapsto (v, 0)$, $w \mapsto (0, w)$ is a coproduct.

Example 5.6. In $\mathbf{Grp}$, the free product $G * H$ (not the direct product) is a coproduct.

Proposition 5.7. *Coproducts are unique up to a unique isomorphism commuting with the injections.*