LECTURE 3. HOMOMORPHISMS, QUOTIENT GROUPS

Please consider every proposition in this text as an exercise.

1. Homomorphisms of Groups

Definition 1.1. Let G and H be groups. A map $\alpha : G \to H$ is called a homomorphism from G to H if

$$\alpha(xy) = \alpha(x) \cdot \alpha(y), \quad \forall x, y \in G. \tag{1.1}$$

The set of all homomorphisms from the group G to the group H is denoted by $\mathrm{Hom}(G, H)$.

Proposition 1.2. *Let $\alpha : G \to H$ be a group homomorphism. Then*

$$\alpha(e_G) = e_H, \tag{1.2}$$

and

$$\alpha(x^{-1}) = (\alpha(x))^{-1}, \quad \forall x \in G. \tag{1.3}$$

Definition 1.3. We are going to use the following terms and notation:

(1) A homomorphism from a group to itself is called an endomorphism. The set of all endomorphisms of the group G is denoted by $\mathrm{End}(G)$.
(2) The map $\mathrm{Id} \in \mathrm{End}(G)$ defined by
$$\mathrm{Id}(x) := x, \quad \forall x \in G, \tag{1.4}$$
is called the identity map.
(3) A homomorphism is called an isomorphism if it is bijective. The set of all isomorphisms from the group G to the group H is denoted by $\mathrm{Iso}(G, H)$. If there exists an isomorphism from G to H, we write $G \cong H$ and say that G and H are isomorphic.
(4) An endomorphism is called an automorphism if it is bijective. The set of all automorphisms of a group G is denoted by $\mathrm{Aut}(G)$.

Proposition 1.4. *Being isomorphic is an equivalence relation on the set of groups.*

Exercise 1.5. Is $\mathrm{End}(G)$ a monoid or a group with respect to the operation of composition of maps? Is $\mathrm{Aut}(G)$ a monoid or a group with respect to the operation of composition?

Definition 1.6. For an arbitrary $\alpha \in \mathrm{Hom}(G, H)$ we define

$$\mathrm{Ker}(\alpha) := \{x \in G \mid \alpha(x) = e_H\}, \quad \mathrm{Im}(\alpha) = \alpha(G) := \{\alpha(x) \mid x \in G\}. \tag{1.5}$$

The set $\mathrm{Ker}(\alpha)$ is called the kernel of α. The set $\alpha(G)$ is called the image of α.

Proposition 1.7. *Let $\alpha : G \to H$ be a group homomorphism. Then* $\mathrm{Ker}(\alpha)$ *is a subgroup of G, and* $\mathrm{Im}(\alpha)$ *is a subgroup of H.*

Proposition 1.8. *Let $\alpha : G \to H$ be a group homomorphism. If* $\mathrm{ord(x)}$ *is finite for $x \in G$, then*

$$\mathrm{ord}(\alpha(\mathrm{x}))|\mathrm{ord(x)}. \tag{1.6}$$

If G is finite, then the set $\alpha(G)$ is also finite, and

$$\mathrm{ord}(\alpha(\mathrm{x}))||\alpha(\mathrm{G})|. \tag{1.7}$$

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Proposition 1.9. *Suppose that K is a subgroup of a group G and $\alpha : G \to H$ is a group homomorphism. Then $\alpha(K)$ is a subgroup of H. If K is cyclic, then $\alpha(K)$ is cyclic. If K is abelian, then $\alpha(K)$ is abelian.*

Proposition 1.10. *Suppose that K is a subgroup of a group H and $\alpha : G \to H$ is a homomorphism of groups. Then the preimage $\alpha^{-1}(K)$ is a subgroup of G.*

Proposition 1.11. *Let $\alpha : G \to H$ be a group homomorphism. Then*

(1)

$$(\alpha \text{ is injective}) \iff (\mathrm{Ker}(\alpha) = \{e_G\}); \tag{1.8}$$

(2)

$$(\alpha \text{ is surjective}) \iff (\alpha(G) = H); \tag{1.9}$$

(3)

$$(\alpha \in \mathrm{Iso}(G, H)) \iff ((\mathrm{Ker}(\alpha) = \{e_G\}) \wedge (\alpha(G) = H)). \tag{1.10}$$

2. Normal Subgroups

Definition 2.1. For an arbitrary element x of a group G define a map $\psi_x : G \to G$ by

$$\psi_x(y) := xyx^{-1}. \tag{2.1}$$

The map ψ_x is called an inner automorphism (of the group G). The set of inner automorphisms of the group G is denoted by $\mathrm{InnAut}(G)$.

Proposition 2.2. *Every inner automorphism of the group G is an automorphism of G. The set* $\mathrm{InnAut}(G)$ *is a subgroup of* $\mathrm{Aut}(G)$.

Exercise 2.3. Define $F : G \to \mathrm{Aut}(G)$ by

$$F(x) := \psi_x. \tag{2.2}$$

Show that F is a homomorphism. Describe $\mathrm{Ker}F$.

Definition 2.4. A subgroup H of a group G is called normal if

$$\psi_x(H) \subset H, \quad \forall x \in G. \tag{2.3}$$

In this case we write $H \lhd G$.

Proposition 2.5. *Let H be a subgroup of a group G. Then*

$$(H \lhd G) \iff (xH = Hx, \forall x \in G). \tag{2.4}$$

Exercise 2.6. Give an example of a subgroup that is not normal.

Proposition 2.7. *Let H be a subgroup of a group G. Denote by $p : G \to G/H$ the canonical projection of G to the set of left cosets G/H, by $p \times p : G \times G \to G/H \times G/H$ the map given by*

$$(p \times p)(x, y) := (p(x), p(y)), \tag{2.5}$$

and by $mult : G \times G \to G$ the multiplication operation in G:

$$mult(x, y) := xy. \tag{2.6}$$

Then there exists a unique map $mult' : G/H \times G/H \to G/H$ such that

$$mult' \circ (p \times p) = p \circ mult \tag{2.7}$$

if and only if $H \lhd G$.

The equality 2.7 can be written as the following commutative diagram below

$$\begin{array}{ccc} G \times G & \xrightarrow{mult} & G \\ \downarrow{\scriptstyle p\times p} & & \downarrow{\scriptstyle p} \\ G/H \times G/H & \xrightarrow{mult'} & G/H \end{array} \tag{2.8}$$

If $H \lhd G$, then the set of left cosets G/H with respect to the operation $mult'$ is a group.

Definition 2.8. Let H be a normal subgroup of a group G. Then the set of left cosets G/H with respect to the operation $mult'$ defined in Proposition 2.7 is called the quotient group G/H.

Note that if $H \lhd G$, then the set of right cosets coincides with the set of left cosets and the multiplication defined similarly to Proposition 2.7 coincides with the multiplication operation in G/H.

Proposition 2.9. *Let $H \lhd G$. Then the canonical projection $p : G \to G/H$ is a surjective group homomorphism.*

Proposition 2.10. *Let $\alpha : G \to H$ be a group homomorphism. Then*

$$\mathrm{Ker}(\alpha) \lhd G. \tag{2.9}$$

Exercise 2.11. Let $\alpha : G \to H$ be a group homomorphism. Is it always true that

$$\mathrm{Im}(\alpha) \lhd H? \tag{2.10}$$

Exercise 2.12. Suppose that $\alpha : G \to H$ is a group homomorphism and $K \lhd G$. Is it always true that

$$\alpha(K) \lhd H? \tag{2.11}$$

Exercise 2.13. Suppose that $\alpha : G \to H$ is a group homomorphism and $K \lhd H$. Is it always true that

$$\alpha^{-1}(K) \lhd G? \tag{2.12}$$

Exercise 2.14. Let G be a group. Is it always true that

$$\mathrm{InnAut}(G) \lhd \mathrm{Aut}(G)? \tag{2.13}$$

Relevant textbook chapters:

- Pinter, Chapters 14, 15;
- Gallian, Chapters 9, 10;
- Fraleigh, Chapter 3;
- Dummit, Foote, Section 3.1;
- Allenby, Sections 5.10, 5.11.