LECTURE 4. ISOMORPHISM THEOREMS

Please consider every proposition in this text as an exercise.

1. First Isomorphism Theorem

Proposition 1.1. *Let $\alpha : G \to H$ be a homomorphism of groups. Then there exists an isomorphism of groups*

$$G/\mathrm{Ker}(\alpha) \cong \mathrm{Im}(\alpha), \tag{1.1}$$

which coincides as a map with the canonical bijection between the sets

$$G/\sim_\alpha \text{ and } \alpha(G). \tag{1.2}$$

Proposition 1.2. *Suppose that G is a group and $g \in G$. Define a map $l_g : G \to G$ by*

$$l_g(x) := gx. \tag{1.3}$$

Then l_g is a bijection for every $g \in G$.

Proposition 1.3 (Cayley)**.** *Let G be a finite group. Then there exists a subgroup H of a symmetric group $S_{|G|}$ such that G and H are isomorphic.*

2. Second and Third Isomorphism Theorems

Definition 2.1. Let M and S be subsets of a group G. Denote by MS the following subset of G:

$$MS := \{ms \mid m \in M, s \in S\}. \tag{2.1}$$

Proposition 2.2. *Suppose that G is a group, K and H are its subgroups and $H \lhd G$. Then HK is a subgroup of G, $H \lhd HK$, $H \cap K \lhd K$ and*

$$HK/H \cong K/H \cap K. \tag{2.2}$$

Proposition 2.3. *Suppose that G is a group, K and H are its normal subgroups and $K \subset H$. Then $H/K \lhd G/K$ and*

$$(G/K)/(H/K) \cong G/H. \tag{2.3}$$

Relevant textbook chapters:

- Pinter, Chapters 9, 16;
- Gallian, Chapters 6, 10;
- Fraleigh, Chapter 3;
- Dummit, Foote, Section 3.3;
- Allenby, Section 5.11.

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