LECTURE 5. GROUP ACTIONS

Please consider every proposition in this text as an exercise.

1. Definitions

Definition 1.1. Suppose that G is a group and X is a set. A map

$$\alpha : G \times X \to X \tag{1.1}$$

is called an action of the group G on the set X if the following conditions hold (we denote the element $\alpha(g, x)$ by gx):

(1) $(g_1 g_2)x = g_1(g_2 x), \quad \forall g_1, g_2 \in G, \forall x \in X;$
(2) $ex = x \quad \forall x \in X.$

For an action of a group G on a set X and an arbitrary $x \in X$ define an orbit to be the following set:

$$\mathrm{Orb}(x) := \{gx \mid g \in G\}. \tag{1.2}$$

The set

$$\mathrm{Stab}(x) := \{g \in G \mid gx = x\} \tag{1.3}$$

is called the stabilizer of the element x.

Definition 1.2. For every set X denote by $S(X)$ the set of bijections from X to itself. Recall that $S(X)$ is a group with respect to the operation of composition.

Proposition 1.3. *Let α be an action of a group G on a set X. For every $g \in G$ define the map $\alpha_g : X \to X$ by*

$$\alpha_g(x) := \alpha(g, x). \tag{1.4}$$

Then

$$\alpha_g \in S(x), \quad \forall g \in G. \tag{1.5}$$

Define a map $\varphi : G \to S(X)$ by

$$\varphi(g) := \alpha_g. \tag{1.6}$$

Then $\varphi \in \mathrm{Hom}(G, S(X))$.

Proposition 1.4. *Suppose that G is a group, X is a set and $\varphi \in \mathrm{Hom}(G, S(X))$. Define a map $\alpha : G \times X \to X$ by*

$$\alpha(g, x) := \varphi(g)(x). \tag{1.7}$$

Then α is an action of the group G on the set X.

Proposition 1.5. *Let α be an action of a group G on a set X. Then $\mathrm{Stab}(x)$ is a subgroup of G and*

$$|\mathrm{Orb}(x)| = \frac{|G|}{|\mathrm{Stab}(x)|} \tag{1.8}$$

Exercise 1.6. Is $\mathrm{Stab}(x)$ always a normal subgroup of G?

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Proposition 1.7. *Let α be an action of a group G on a set X. Define*

$$(x \sim y) \iff (\exists g \in G\, gx = y). \tag{1.9}$$

Then it is an equivalence relation and the equivalence classes are exactly orbits of the action α. We get that

$$X = \bigsqcup_M M, \tag{1.10}$$

where the union is taken over the set of orbits. If the set X is finite, then we get that

$$|X| = \sum_M |M|, \tag{1.11}$$

where the sum is taken over the set of orbits.

Exercise 1.8. Suppose that x and y belong to the same orbit of an action of a group G on a set X. Is there a connection between the subgroups $\mathrm{Stab}(x)$ and $\mathrm{Stab}(y)$?

2. Conjugation

Definition 2.1. Suppose that the action of a group G on itself is determined by the homomorphism $\psi : G \to \mathrm{Aut}(G)$ given by

$$\psi(g)(x) := gxg^{-1}. \tag{2.1}$$

The orbits of this action are called conjugacy classes of the group G.

Exercise 2.2. Describe the conjugacy classes of the group S_n.

Exercise 2.3. Describe the conjugacy classes of the group A_4.

Definition 2.4. Let S be a subset of a group G. Define the normalizer of the set S by

$$N_G(S) := \{g \in G \mid \psi(g)(S) \subset S\}. \tag{2.2}$$

Define the centralizer of the set S by

$$C_G(S) := \{g \in G \mid gxg^{-1} = x,\ \forall x \in S\}. \tag{2.3}$$

Proposition 2.5. *For every subset S of a group G both $N_G(S)$ and $C_G(S)$ are subgroups of the group G.*

Proposition 2.6. *Suppose that G is a finite group, and C is a conjugacy class, containing an element x. Then*

$$|C| = \frac{|G|}{|C_g(\{x\})|}. \tag{2.4}$$

Proposition 2.7 (The Class Equation)**.** *Let G be a finite group. Then*

$$|G| = |Z(G)| + \sum_{x \in S} \frac{|G|}{|C_g(\{x\})|}, \tag{2.5}$$

where S is the set of representatives of conjugacy classes of the size greater than one.

Exercise 2.8. Let H be a normal subgroup of a group G. Define $\psi : G \to \mathrm{Aut}(H)$ by

$$\psi(g)(x) := gxg^{-1}. \tag{2.6}$$

Find $\mathrm{Ker}(\psi)$.

Proposition 2.9. *Let H be a subgroup of a group G. Then H is a normal subgroup of the group $N_G(H)$.*

Suppose that K is a subgroup of G such that $H \lhd K$. Then $K \leq N_G(H)$.

Proposition 2.10. *Let H be a subgroup of a group G. Then $C_G(H) \lhd N_G(H)$.*

Relevant textbook chapters:

- Dummit, Foote, Sections 1.7, 4.1-4.3;