LECTURE 6. FREE GROUPS

Please consider every proposition in this text as an exercise.

1. Free Groups

Definition 1.1. Let X be a set. The alphabet associated to X is the disjoint union

$$X^{\pm 1} = X \times \{1, -1\}. \tag{1.1}$$

We write x^1 and x^{-1} for the pairs $(x, 1)$ and $(x, -1)$. Also define

$$(x^\varepsilon)^{-1} = x^{-\varepsilon} \tag{1.2}$$

for $\varepsilon \in \{1, -1\}$.

Definition 1.2. A word on X is a finite sequence $(a_1, a_2, \ldots, a_n)$ with $n \geq 0$ and each $a_i \in X^{\pm 1}$. The empty sequence ($n = 0$) is called the empty word and is denoted e.

The set of all words on X is denoted $W(X)$. The length of a word $w = (a_1, \ldots, a_n)$ is $|w| = n$. Concatenation of words $u = (a_1, \ldots, a_m)$ and $v = (b_1, \ldots, b_n)$ is defined as

$$u \cdot v = (a_1, \ldots, a_m, b_1, \ldots, b_n). \tag{1.3}$$

Exercise 1.3. Is $W(X)$ a group or a monoid?

Definition 1.4. A word admits an elementary reduction if it contains a subword of the form $x^\varepsilon x^{-\varepsilon}$ for some $x \in X$, $\varepsilon \in \{1, -1\}$. Deleting that adjacent pair yields a new word.

A word is reduced if no elementary reduction can be applied, i.e. it contains no adjacent inverse letters $x^\varepsilon x^{-\varepsilon}$.

Proposition 1.5. *Every word w can be transformed into a reduced word by a finite sequence of elementary reductions. Moreover, the resulting reduced word is unique; it is called the reduced form of w and is denoted $\overline{w}$.*

Definition 1.6. Define an equivalence relation on $W(X)$ by $u \sim v \iff \overline{u} = \overline{v}$.

Exercise 1.7. Show that this is indeed an equivalence relation. Denote the equivalence class containing an element u by $[u]$.

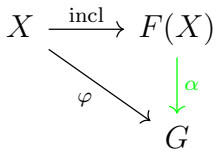
Definition 1.8 (Free group)**.** The free group on X, denoted $F(X)$, is the set of equivalence classes $W(X)/\sim$ endowed with the operation of multiplication defined by

$$[u] \cdot [v] := [u \cdot v]. \tag{1.4}$$

Proposition 1.9. *The multiplication on $F(X)$ is well-defined and $F(X)$ is a group with respect to this operation.*

We identify an element of $F(X)$ with its unique reduced representative. The set X embeds into $F(X)$ via $x \mapsto [x^1]$.

Proposition 1.10 (Universal property of $F(X)$)**.** *Let* $\mathrm{incl} : X \hookrightarrow F(X)$ *be the natural inclusion. For any group G and any set map $\varphi : X \to G$, there exists a unique group homomorphism $\alpha : F(X) \to G$ such that the following diagram commutes:*



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2. Free Abelian Group

Definition 2.1. A function $f : X \to \mathbb{Z}$ has finite support if the set

$$\{x \in X \mid f(x) \neq 0\}$$

is finite.

Proposition 2.2. *Let X be a set. For two finitely supported functions $f, g : X \to \mathbb{Z}$, define their (pointwise) sum by*

$$(f+g)(x) := f(x) + g(x), \qquad x \in X.$$

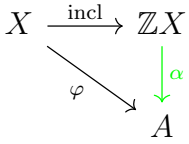
The function $0(x) = 0$ is the identity and $(-f)(x) = -f(x)$ is the inverse of f. This makes the set of all finitely supported functions an abelian group.

Definition 2.3 (Free abelian group)**.** The free abelian group on X, denoted $\mathbb{Z}X$ (sometimes $F_{\mathrm{ab}}(X)$), is the group of all finitely supported functions $f : X \to \mathbb{Z}$ with pointwise addition. Equivalently,

$$\mathbb{Z}X \cong \bigoplus_{x \in X} \mathbb{Z},$$

the direct sum of $|X|$ copies of $\mathbb{Z}$. The set X is identified with the standard basis elements: $x \mapsto \delta_x$ where $\delta_x(y) = 1$ if $y = x$ and 0 otherwise.

Proposition 2.4 (Universal property of $\mathbb{Z}X$)**.** *Let* $\mathrm{incl} : X \hookrightarrow \mathbb{Z}X$ *be the inclusion mapping x to δ_x. For any abelian group A and any set map $\varphi : X \to A$, there exists a unique group homomorphism $\alpha : \mathbb{Z}X \to A$ such that the diagram commutes:*



3. Groups Presented by Generators and Relations

Definition 3.1. Let X be a set and $R \subseteq F(X)$ a set of reduced words. The group presented by $\langle X \mid R \rangle$ is the quotient $F(X)/N$, where N is the normal closure of R in $F(X)$ (the smallest normal subgroup containing R). The elements of X are called generators and the elements of R are called relators. A relation $r = e$ is often written when $r \in R$.

Exercise 3.2. Find some presentations by generators and relations of the groups $\mathbb{Z}$, $\mathbb{Z}_n$, $F(\{x, y\})$ and S_3.

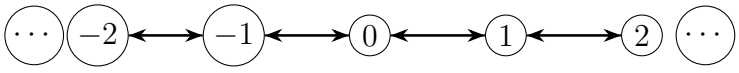
4. Cayley Diagrams

Definition 4.1 (Cayley graph $\Gamma(G, S)$)**.** Let G be a group and $S \subset G$ a generating set not containing the identity. The Cayley diagram (or Cayley graph) of G with respect to S is the directed, edge-coloured graph $\Gamma(G, S)$ defined as follows.

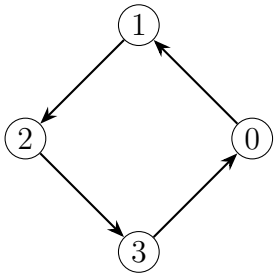
The vertex set is G. For each $g \in G$ and each $s \in S$, there is a directed edge from g to gs, labelled (or coloured) by s. If $s = s^{-1}$, one usually draws an undirected edge with the label s.

In the following diagrams, arrows indicate directed edges and plain lines correspond to involutions.

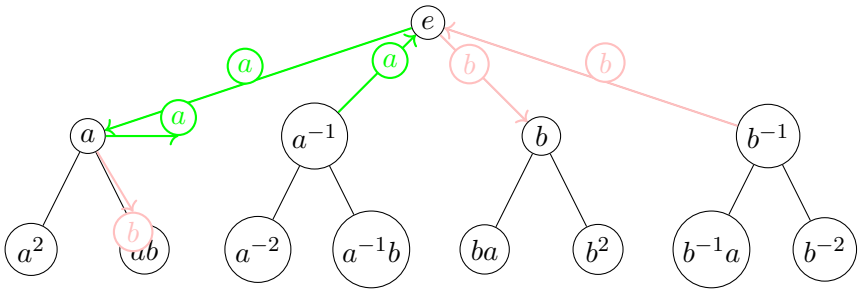
Example 4.2. $\Gamma(\mathbb{Z}, \{1\})$:



$\mathbb{Z}_4 = \langle 1 \mid 4 \cdot 1 = 0 \rangle$ with generator 1 ($-1 = 3$). Its Cayley graph $\Gamma(\mathbb{Z}_4, \{1\})$:

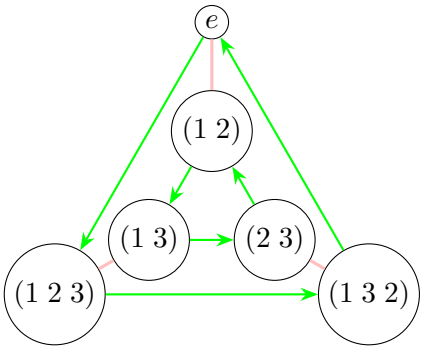


Example 4.3 (Free group on two generators $F(a,b)$). $\Gamma(F(a,b),\{a,b\})$ is a 4-regular infinite tree. A fragment with root e at the top:



Green edges correspond to the generator a, pink edges to b.

Example 4.4 (Symmetric group S_3). S_3 is generated by a 3-cycle $a = (1\ 2\ 3)$ and a transposition $b = (1\ 2)$. Then $a^3 = e$, $b^2 = e$, and $bab = a^{-1}$.



5. Derived Subgroup

Definition 5.1. For a, b in a group G, their commutator is $[a,b] = aba^{-1}b^{-1}$.

Definition 5.2. The derived subgroup (or commutator subgroup) of G, denoted G' or $[G,G]$, is the subgroup generated by all commutators:

$$G' = \langle\, [a,b] \mid a, b \in G \,\rangle.$$

Definition 5.3 (Abelianization)**.** The quotient group G/G' is called the abelianization of G.

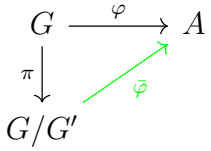
Proposition 5.4. *G' is a normal subgroup of G.*

Proposition 5.5. *G/G' is an abelian group.*

Proposition 5.6. *G' is the smallest normal subgroup with abelian quotient: if $N \trianglelefteq G$ and G/N is abelian, then $G' \subseteq N$.*

Proposition 5.7. *For any homomorphism $\varphi : G \to A$ where A is abelian, φ factors uniquely through the canonical projection $\pi : G \to G/G'$. That is, there exists a unique homomorphism*

$\bar{\varphi} : G/G' \to A$ *such that the following diagram commutes:*



Proposition 5.8. *For any set X, the abelianization of the free group $F(X)$ is isomorphic to the free abelian group on X:*

$$F(X)/F(X)' \cong \mathbb{Z}X.$$

Relevant textbook chapters:

- Dummit, Foote, Section 6.3;
- Gallian, Chapter 25.