LECTURE 7. RINGS AND IDEALS

Please treat every proposition in this text as an exercise.

1. Rings

Definition 1.1. Suppose that R is a non-empty set and two maps $\mathrm{add} : R \times R \to R$ and $\mathrm{mult} : R \times R \to R$ are defined. As usual, we denote $\mathrm{add}(a, b)$ by $a + b$ and $\mathrm{mult}(a, b)$ by ab. The triple $(R, \mathrm{add}, \mathrm{mult})$ is called a rng if the following conditions hold:

(1) (R, add) is an abelian group (we denote its identity element by 0);
(2)
$$a(bc) = (ab)c, \quad \forall a, b, c \in R; \tag{1.1}$$
(3)
$$a(b + c) = ab + ac, \quad \forall a, b, c \in R, \tag{1.2}$$
$$(a + b)c = ab + bc, \quad \forall a, b, c \in R. \tag{1.3}$$

A rng R is called a ring if (R, mult) is a monoid. The multiplication identity element is denoted by 1. A rng or a ring R is called commutative if

$$ab = ba, \quad \forall a, b \in R. \tag{1.4}$$

Exercise 1.2. Let R be a rng. Is it always true that

$$a \cdot 0 = 0 \cdot a = 0, \quad \forall a \in R? \tag{1.5}$$

Exercise 1.3. Suggest an example of a commutative rng that is not a ring.

Next in this lecture, we will consider rings. However, in some cases the corresponding construction or statement makes sense for rngs as well.

Definition 1.4. A non-zero commutative ring R is called an integral domain if

$$\forall a, b \in R \quad (ab = 0) \implies (a = 0 \text{ or } b = 0). \tag{1.6}$$

A non-zero ring R is called a division ring if $(R \setminus \{0\}, \mathrm{mult})$ is a group. A field is a commutative division ring.

For a ring R denote by $\mathrm{char}R$ (the characteristic of the ring R) the minimal positive integer n such that

$$\underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = 0. \tag{1.7}$$

If there is no such positive integer we write $\mathrm{char}R := 0$.

Exercise 1.5. For each of the following sets with the usual operations of addition and multiplication determine which of the terms introduced above apply:

(1) $\mathbb{Z}$;
(2) $\mathbb{R}$;
(3) $\mathrm{Mat}(n \times n, \mathbb{Z})$;
(4) $\mathrm{Mat}(n \times n, \mathbb{R})$.

Exercise 1.6. Suggest an example of a non-zero commutative ring that is not an integral domain.

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Exercise 1.7. Let R be an integral domain, $a, b \in R$. Can the equation

$$ax = b \tag{1.8}$$

have more than one solution? Can it have no solutions?

Exercise 1.8. Suggest an example of a ring of non-zero characteristic.

Definition 1.9. A subset K of a ring $(R, +, \cdot)$ is called a subring if it is a ring with respect to the addition and multiplication inherited from R.

Exercise 1.10. Is Definition 1.9 equivalent to K satisfying the following two conditions:

(1) $a + b \in K$, $\forall a, b \in K$;
(2) $ab \in K$, $\forall a, b \in K$?

Exercise 1.11. Let K be a subring of a ring $(R, +, \cdot)$. Then $(K, +)$ is a normal subgroup of $(R, +)$. Suggest an example in which the multiplication operation of R cannot be transferred to R/K, i.e. there are two pairs $a, b \in R$ and $c, d \in R$ of representatives of two elements of R/K such that the products ac and bd belong to different elements of R/K.

2. Ideals

Exercise 1.11 inspires the following definition.

Definition 2.1. A subset I of a ring $(R, +, \cdot)$ is called a left ideal of R if $(I, +)$ is a subgroup of $(R, +)$ and $rx \in I$ for all $r \in R, x \in I$. A subset I of a ring $(R, +, \cdot)$ is called a right ideal of R if $(I, +)$ is a subgroup of $(R, +)$ and $xr \in I$ for all $r \in R, x \in I$. A two-sided ideal is a left ideal that is also a right ideal.

Proposition 2.2. *Let R be a ring and let I and J be left (right, two-sided) ideals of R. Then*

$$I \cap J \qquad \text{and} \qquad I + J := \{ x + y \mid x \in I,\ y \in J \} \tag{2.1}$$

are ideals of R of the same type.

Exercise 2.3. Give an example of a left ideal of some ring that is not a two-sided ideal.

Definition 2.4. Let R and K be rings. A map $\varphi : R \to K$ is called a ring homomorphism if the following conditions hold:

(1)
$$\varphi(a + b) = \varphi(a) + \varphi(b), \quad \forall a, b \in R; \tag{2.2}$$

(2)
$$\varphi(ab) = \varphi(a) \cdot \varphi(b), \quad \forall a, b \in R; \tag{2.3}$$

(3)
$$\varphi(1_R) = 1_K. \tag{2.4}$$

Define

$$\mathrm{Ker}(\varphi) := \{x \in R \mid \varphi(x) = 0_K\}, \quad \mathrm{Im}(\varphi) = \varphi(R) := \{\varphi(x) \mid x \in R\}. \tag{2.5}$$

The set $\mathrm{Ker}(\varphi)$ is called the kernel of φ. The set $\varphi(R)$ is called the image of φ.

A homomorphism is called an isomorphism if it is bijective.

Exercise 2.5. Find a ring homomorphism whose image is not an ideal.

Proposition 2.6. *Let I be a two-sided ideal of a ring R. Let $p : R \to R/I$ denote the canonical projection of the group $(R, +)$ onto the quotient group $(R/I, +)$. Then there exists a unique map* $\text{mult}' : R/I \times R/I \to R/I$ *such that*

$$\text{mult}' \circ (p \times p) = p \circ \text{mult}. \tag{2.6}$$

The equality 2.6 can be written as the following commutative diagram

$$\begin{array}{ccc} R \times R & \xrightarrow{\text{mult}} & R \\ \downarrow{\scriptstyle p\times p} & & \downarrow{\scriptstyle p} \\ R/I \times R/I & \xrightarrow{\text{mult}'} & R/I. \end{array} \tag{2.7}$$

Then $(R/I, +, \text{mult}')$ is a ring. It is called a quotient ring. Also, the map p is a ring homomorphism and

$$\text{Ker}(p) = I. \tag{2.8}$$

Proposition 2.7. *Let φ be a homomorphism between rings R and K. Then:*

(1) $\text{Ker}(\varphi)$ *is a two-sided ideal of R;*
(2) $\text{Im}(\varphi)$ *is a subring of K;*
(3) the ring $R/\text{Ker}(\varphi)$ *is naturally isomorphic to the ring* $\text{Im}(\varphi)$.

Definition 2.8. An ideal I of a commutative ring R is called prime if it satisfies the following two conditions:

(1)
$$\forall a, b \in R \quad (ab \in I) \implies ((a \in I) \text{ or } (b \in I)); \tag{2.9}$$
(2) $I \neq R$.

Proposition 2.9. *Let I be an ideal of a commutative ring R. Then*

$$(I \text{ is prime}) \iff (\text{The ring } R/I \text{ is an integral domain}). \tag{2.10}$$

Definition 2.10. A two-sided ideal I of a ring R is called maximal if it satisfies the following two conditions:

(1)
$$\text{For every ideal } J \text{ of the ring } R \quad (I \subset J) \implies ((J = I) \text{ or } (J = R)); \tag{2.11}$$
(2) $I \neq R$.

Proposition 2.11. *Let I be an ideal of a commutative ring R. Then*

$$(I \text{ is maximal}) \iff (\text{The ring } R/I \text{ is a field}). \tag{2.12}$$

Exercise 2.12. Is it true that in a commutative ring every maximal ideal is a prime ideal?

Proposition 2.13. *The ring $\mathbb{Z}_n$ is a field if and only if n is a prime number.*

Proposition 2.14. *The characteristic of an integral domain is either 0 or a prime number.*

Proposition 2.15. *Every finite integral domain is a field.*

Relevant textbook chapters:

- Dummit, Foote, Sections 7.1-7.4;
- Pinter, Ch. 17-20;
- Gallian, Ch. 12-15;
- Fraleigh, Ch. 4, 5.