PROBLEM SET 1

1. Divisibility of Integers

Problem 1.1. *Show that*

$$\sqrt{p} \notin \mathbb{Q} \tag{1.1}$$

for every prime number p.

Problem 1.2. *Let* $\mathrm{GCD(m, n)} = \mathrm{d}$. *Is the solution* $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ *of the equation*

$$mx + ny = d \tag{1.2}$$

unique?

Problem 1.3. *Suppose that*

$$mx + ny = 1, \quad m, n, x, y \in \mathbb{Z} \setminus \{0\} \tag{1.3}$$

Does it imply that

$$\mathrm{GCD(m, n)} = 1? \tag{1.4}$$

Can we claim that

$$\mathrm{GCD(mx, ny)} = 1? \tag{1.5}$$

Can we claim that

$$\mathrm{GCD(mn, xy)} = 1? \tag{1.6}$$

Problem 1.4. *Suppose that* $m, k \geq 2$ *and* $m^k - 1$ *is a prime number. Show that* k *is also a prime number.*

Problem 1.5. *Compute the remainder when* 2^{2026} *is divided by* 7.

Problem 1.6. *Suppose that* $\mathrm{GCD(k, n)} = 1$. *Show that* $\exists m \in \mathbb{Z}$ *such that*

$$mk \equiv 1 \pmod{n} \tag{1.7}$$

Problem 1.7. *For which* $k \in \mathbb{Z}$ *does the equation*

$$x^2 \equiv k \pmod{4} \tag{1.8}$$

have solutions?

Problem 1.8. *Is it true that* $12|(n^4 - n^2)$, $\forall n \in \mathbb{Z}$*?*

Problem 1.9. *Does the equation*

$$a^2 + b^2 = 3c^2 \tag{1.9}$$

have non-zero integer solutions?

Problem 1.10. *Let* $n \in \mathbb{Z}_+$, *define the Euler* φ*-function* $\varphi(n)$ *by*

$$\varphi(n) := |\{1 \leq k \leq n \mid \mathrm{GCD(k, n)} = 1\}|. \tag{1.10}$$

Show that if $\mathrm{GCD(m, n)} = 1$, *then*

$$\varphi(mn) = \varphi(m)\varphi(n). \tag{1.11}$$

Problem 1.11. *Suggest a formula for* $\varphi(n)$ *using the factorization of* n *into primes.*

Problem 1.12. *Show that*

$$\lim_{n \to +\infty} \varphi(n) = +\infty. \tag{1.12}$$

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