PROBLEM SET 2

2. Groups and Subgroups

Problem 2.1. *Which of the following subsets of matrices is a group with respect to the standard matrix multiplication:*

(1)
$$\mathrm{Mat}(n \times n, \mathbb{R}), \tag{2.1}$$

(2)
$$GL_n(\mathbb{R}) := \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid \det A \neq 0\}, \tag{2.2}$$

(3)
$$SL_n(\mathbb{R}) := \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid \det A = 1\}, \tag{2.3}$$

(4)
$$\{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid \det A = 0\}, \tag{2.4}$$

(5)
$$O_n(\mathbb{R}) := \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid A \cdot A^t = \mathrm{Id}\}, \tag{2.5}$$

(6)
$$SO_n(\mathbb{R}) := \{A \in \mathrm{Mat}(n \times n, \mathbb{R}) \mid A \cdot A^t = \mathrm{Id},\ \det A = 1\}? \tag{2.6}$$

Problem 2.2. *Suppose that the order of every non-identity element of a group G equals 2. Show that the group G is abelian.*

Problem 2.3. *Define the center $Z(G)$ of a group G as*
$$Z(G) := \{x \in G \mid xy = yx\ \forall y \in G\}. \tag{2.7}$$
Show that $Z(G)$ is a subgroup of G. Find $Z(G)$ when G is the group of all bijections from a set M to itself.

Problem 2.4. *Suppose that S is a non-empty subset of a finite group G such that $\forall a, b \in S$ we have that $ab \in S$. Is it always true that S is a subgroup G?*

Problem 2.5. *Let a and b be elements of a group G such that* ord(ab) *is finite. Show that* ord(ba) *is also finite and*
$$\mathrm{ord(ab)} = \mathrm{ord(ba)}. \tag{2.8}$$

Problem 2.6. *Suppose that G is a group, a, b are elements of finite order, and $ab = ba$. Show that*
$$(\mathrm{ord(ab)} = \mathrm{ord(a)} \cdot \mathrm{ord(b)}) \iff (\mathrm{GCD}(\mathrm{ord(a)}, \mathrm{ord(b)}) = 1)\,. \tag{2.9}$$

Problem 2.7. *Is it always true that a subgroup of a cyclic group is cyclic?*

Problem 2.8. *Suppose that G is a finite cyclic group and d is a divisor of $|G|$. Show that there exists exactly one subgroup of G of order d.*

Problem 2.9. *Let p and q be distinct prime numbers. Show that every abelian group of order pq is cyclic.*

Problem 2.10. *Suppose that a group G has only two subgroups: $\{e\}$ and G. Show that G is a finite cyclic group of prime order.*

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Problem 2.11. *Let the map* $\mathrm{inv} : \mathrm{G} \to \mathrm{G}$ *be defined by*

$$\mathrm{inv}(\mathrm{x}) := \mathrm{x}^{-1}. \tag{2.10}$$

Let H *be a subgroup of* G*. Show that* inv *determines a bijection between the set of left cosets* G/H *and right cosets* $H\backslash G$*.*

Problem 2.12. *Suppose that* K *is a subgroup of a finite group* G*, and* H *is a subgroup of* K*. Show that* H *is a subgroup of* G *and*

$$|G/H| = |G/K| \cdot |K/H|. \tag{2.11}$$

Problem 2.13. *Show that every group of an even order has an element of order* 2*.*

Problem 2.14. *Let* G *be a finite group. Define*

$$S := \{x \in G \mid x^5 = e\}. \tag{2.12}$$

Is it always true that $4|(|S| - 1)$*?*