PROBLEM SET 3

3. Homomorphisms, Quotient Groups

Problem 3.1. *Describe the group* $\mathrm{Aut}(\mathbb{Z})$.

Problem 3.2. *Describe the group* $\mathrm{Aut}(\mathbb{Z}_n)$.

Problem 3.3. *Let* $\mathrm{Aut}(\mathbb{Z}_n) = \mathrm{Aut}(\mathbb{Z}_m)$. *Is it always true that* $m = n$*?*

Problem 3.4. *Does there exist a group isomorphic to its subgroup?*

Definition 3.5. A map $\rho : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is called a permutation (of the set $\{1, 2, \dots, n\}$) if ρ is a bijection. The set of all permutations of the set $\{1, 2, \dots, n\}$ is called a symmetric group (of n-th order) and is denoted by S_n. The composition $\rho \circ \sigma$ of two permutations $\rho, \sigma \in S_n$ is denoted by $\rho\sigma$. For an element $\rho \in S_n$ we may use a two-row notation:

$$\rho = \begin{pmatrix} 1 & 2 & \dots & n \\ \rho(1) & \rho(2) & \dots & \rho(n). \end{pmatrix} \tag{3.1}$$

A more common notation is cycle notation. It is defined in the following way. Let us fix an arbitrary permutation $\rho \in S_n$. We introduce an equivalence relation on the set $\{1, 2, \dots, n\}$ by the following rule:

$$(i \sim j) \quad \Longleftrightarrow \quad (\exists k \in \mathbb{Z}_+ \text{ such that } \rho^k(i) = j), \tag{3.2}$$

where

$$\rho^k := \underbrace{\rho \circ \rho \circ \dots \circ \rho}_{k \text{ times}}. \tag{3.3}$$

One may check that it is indeed an equivalence relation. For each equivalence class that consists of, say, m elements, we pick a representative j and write a cycle $(j, \rho(j), \rho^2(j), \dots, \rho^{m-1}(j))$ in the cycle notation of ρ. If $m = 1$, we usually omit the corresponding one-element cycle. Here is an example of cycle notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 1 & 4 & 3 \end{pmatrix} = (1, 5, 4)(3, 6). \tag{3.4}$$

From the equivalence relation definition it follows that cycle notation is well-defined. Therefore, every permutation can be represented as a product of disjoint cycles. A cycle of length 2 is called a transposition.

Problem 3.6. *Show that* S_n *is a group with respect to composition and find its order.*

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Problem 3.7. *Show that for every permutation $\rho \in S_n$ and for every subset $\{i_1, i_2, \dots, i_k\}$ of the set $\{1, 2, \dots, n\}$ the following equality holds*

$$\boxed{\rho \circ (i_1, i_2, \dots, i_k) \circ \rho^{-1} = (\rho(i_1), \rho(i_2), \dots, \rho(i_k)),} \tag{3.5}$$

, where the LHS is the composition of three permutations such that the second permutation is a cycle of length k.

Problem 3.8. *Show that every permutation $\rho \in S_n$ can be represented as a product of some transpositions.*

Problem 3.9. *For every positive integer n there exists a unique map $\operatorname{sgn} : S_n \to \{1, -1\}$ such that*

(1) $\operatorname{sgn}((i, j)) = -1, \quad \forall 1 \leq i < j \leq n;$
(2) $\operatorname{sgn}(\rho\sigma) = \operatorname{sgn}(\rho)\operatorname{sgn}(\sigma).$

The value $\operatorname{sgn}(\rho)$ is called the sign of the permutation ρ. The kernel of the map $\operatorname{sgn} : S_n \to \{1, -1\}$ is called the alternating group (of n-th order) and is denoted by A_n.

Problem 3.10. *For which n is it true that $A_n \lhd S_n$?*

Problem 3.11. *Let us assume that the permutations from the group S_n are defined on the element $(n+1)$, mapping it to itself. Then we can consider S_n as a subset of S_{n+1}. For which n is it true that $S_n \lhd S_{n+1}$?*

Problem 3.12. *Is $\mathrm{SL}_n(\mathbb{R})$ a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$? Is $\mathrm{SO}_n(\mathbb{R})$ a normal subgroup of $\mathrm{SL}_n(\mathbb{R})$?*

Problem 3.13. *Let*

$$U := \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid ac \neq 0 \right\} \tag{3.6}$$

Is H a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$?

Problem 3.14. *Is it always true that the quotient group of a cyclic group is cyclic?*

Problem 3.15. *Is it always true that the quotient group of an abelian group is abelian?*

Problem 3.16. *Suppose that $H \lhd G$, H and G/H are abelian. Is it always true that G is also abelian?*

Problem 3.17. *Suppose that $K \lhd H$ and $H \lhd G$. Is it always true that $K \lhd G$?*

Problem 3.18. *Suppose that H is a subgroup of a group G and*

$$|G/H| = 2. \tag{3.7}$$

Is it always true that H is a normal subgroup of G?