PROBLEM SET 4

4. Isomorphism Theorems

Problem 4.1. *Let G and H be groups. Define the operation of multiplication on the set $G \times H$ by*

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 g_2, h_1 h_2). \tag{4.1}$$

Show that $G \times H$ is a group with respect to this operation. It is called the (external) direct product of the groups G and H. Besides the notation $G \times H$, it may be denoted $G \oplus H$, especially, if both G and H are abelian.

Problem 4.2. *Suppose that H and K are normal subgroups of a group G, and $H \cap K = \{e\}$. We already know that HK is a subgroup of G. Show that the natural map from $H \times K$ to HK is an isomorphism of groups. In this case, it is natural to say that the subgroup HK is the (internal) direct product of the subgroups H and K.*

Problem 4.3. *Let G and H be groups. Define the projection maps $p_G : G \times H \to G$ and $p_H : G \times H \to H$ by*

$$p_G(g, h) := g; \tag{4.2}$$

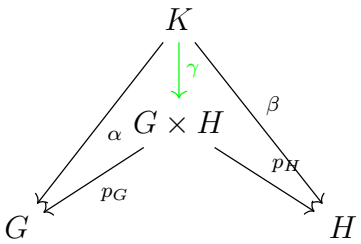
$$p_H(g, h) := h. \tag{4.3}$$

Show that p_G and p_H are homomorphisms.

Suppose that there is a group K and homomorphisms $\alpha : K \to G$ and $\beta : K \to H$. Then there exists a unique homomorphism $\gamma : K \to G \times H$ such that

$$p_G \circ \gamma = \alpha; \tag{4.4}$$

$$p_H \circ \gamma = \beta. \tag{4.5}$$



(4.6)

Problem 4.4. *Show that every subgroup of a quotient group G/K is equal to the group H/K, where H is a subgroup of G.*

Problem 4.5. *Does there exist a surjective homomorphism from $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$?*

Problem 4.6. *Does there exist a surjective homomorphism from $\mathbb{Z}_{16}$ to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$?*

Problem 4.7. *How many homomorphisms are there from the group $\mathbb{Z}_{20}$ to the group $\mathbb{Z}_8$?*

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Problem 4.8. *Let p be a prime number. How many homomorphisms are there from the group $\mathbb{Z}_p \oplus \mathbb{Z}_p$ to the group $\mathbb{Z}_p$?*

Problem 4.9. *For which positive integers m and n is the group $\mathbb{Z}_m \oplus \mathbb{Z}_n$ isomorphic to the group $\mathbb{Z}_{mn}$?*

Problem 4.10 (Chinese remainder theorem)**.** *Suppose that $m_1, \ldots, m_k$ are positive integers such that*

$$\mathrm{GCD}(\mathrm{m_i}, \mathrm{m_j}) = 1, \quad \forall 1 \leq \mathrm{i} \neq \mathrm{j} \leq \mathrm{k}. \tag{4.7}$$

Define a map

$$\phi : \mathbb{Z}_{m_1 m_2 \ldots m_k} \to \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_k} \tag{4.8}$$

by

$$\phi(a) := (a \pmod{m_1}, a \pmod{m_2}, \ldots a \pmod{m_k}) \tag{4.9}$$

Show that the map ϕ is well-defined and is an isomorphism of the groups $\mathbb{Z}_{m_1 m_2 \ldots m_k}$ and $\mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_k}$.

Problem 4.11. *Describe all homomorphisms from the group S_4 to the group $\mathbb{Z}_2$.*

Problem 4.12. *Describe all homomorphisms from the group S_4 to the group $\mathbb{Z}_3$.*

Problem 4.13. *Show that the set*

$$K_4 := \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \tag{4.10}$$

is a normal subgroup of S_4. Construct an isomorphism between S_4/K_4 and S_3.