PROBLEM SET 5

5. Group Actions

Definition 5.1. A map from a Euclidean space $\mathbb{R}^n$ to itself is called an isometry if it preserves the distance between any two points. Recall from Linear Algebra that if an isometry preserves the origin then it is a linear map and, moreover, an orthogonal operator.

Definition 5.2. Let $S \subset \mathbb{R}^n$. An isometry α of $\mathbb{R}^n$ is called an isometry of the set S if

$$\alpha(S) = S. \tag{5.1}$$

Problem 5.3. *Let $S \subset \mathbb{R}^n$. Show that the set of isometries of S is a subgroup of the group of isometries of $\mathbb{R}^n$.*

Problem 5.4. *Show that the group of isometries of the regular tetrahedron is isomorphic to S_4.*

Problem 5.5. *Show that the group of isometries of the cube is isomorphic to $S_4 \times \mathbb{Z}_2$.*

Problem 5.6. *Show that the group of isometries of the regular octahedron is isomorphic to the group of isometries of the cube.*

Problem 5.7. *Let $H \leq G$. Show that the group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of* $\mathrm{Aut}(H)$.

Problem 5.8. *Suppose that the group of inner automorphisms of a group G is cyclic. Show that G is abelian.*

Problem 5.9. *Suppose that G is a non-abelian group of order pq, where p and q are prime numbers. Is it always true that $Z(G) = \{e\}$?*

Problem 5.10. *Suppose that G is a group of order p^k, where p is a prime number, $k \in \mathbb{Z}_+$. Show that $Z(G) \neq \{e\}$.*

Problem 5.11. *Does there exist a non-abelian group of order p^2, where p is a prime number?*

Problem 5.12. *Suppose that G is an abelian group, p is a prime number, and p divides $|G|$. Show that there exists a subgroup of G of order p.*

Problem 5.13 (Cauchy's theorem)**.** *Suppose that G is a group, p is a prime number, and p divides $|G|$. Show that there exists a subgroup of G of order p.*

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Problem 5.14. *Suppose that G is a group of order pq, p and q are prime numbers, $p < q$, and p does not divide $(q-1)$. Show that G is abelian.*

Problem 5.15. *Suppose that G and H are groups and $\psi \in \mathrm{Hom}(G, \mathrm{Aut}(H))$. Define the operation of multiplication on the set $G \times H$ by*

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 g_2, h_1 \psi(g_1)(h_2)). \tag{5.2}$$

The formula (5.2) is motivated by the equality

$$h_1 g_1 h_2 g_2 = h_1 (g_1 h_2 g_1^{-1}) g_1 g_2, \tag{5.3}$$

which would hold if g_1, g_2, h_1, h_2 were elements of the same group.

Show that $G \times H$ is a group with respect to this operation. It is called the (outer) semidirect product of the groups G and H (associated with the homomorphism ψ) and is denoted by $G \ltimes_\psi H$, or, by $G \ltimes H$, by $H \rtimes_\psi G$, or by $H \rtimes G$.

Problem 5.16. *Suppose that H and K are subgroups of a group G, $H \lhd G$ and $H \cap K = \{e\}$. We already know that HK is a subgroup of G. Show that the group HK is isomorphic to a semidirect product $H \rtimes K$. In this case, it is natural to say that the subgroup HK is an (internal) semidirect product of the subgroups K and H.*

Problem 5.17. *Is it true that the group S_3 is isomorphic to a semidirect product of the groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$?*

Problem 5.18. *Can the group A_4 be represented as a semidirect product of two of its proper subgroups?*

Problem 5.19. *Suppose that G is a non-abelian group of order pq, p and q are prime numbers and $p|(q-1)$. Show that every non-abelian group is isomorphic to a semidirect product $\mathbb{Z}_3 \ltimes \mathbb{Z}_7$.*

Definition 5.20. Let K and H be groups. A group G is called an extension of K by H if there exists a subgroup $H' \leq G$ such that the following conditions hold:

(1) $H' \lhd G$;
(2) $H' \cong H$;
(3) $G/H' \cong K$.

Problem 5.21. *Classify all extensions of $\mathbb{Z}_2$ by $\mathbb{Z}_5$ up to isomorphism.*