PROBLEM SET 6

6. Free Groups

Definition 6.1. Let $n \geq 3$. The group of isometries of a regular n-sided polygon is called the dihedral group D_n.

Problem 6.2. *Show that*

$$|D_n| = 2n. \tag{6.1}$$

Problem 6.3. *Show that*

$$D_n \cong \langle a, s \mid a^n = s^2 = sasa = e \rangle. \tag{6.2}$$

Problem 6.4. *Describe the conjugacy classes of the group D_n.*

Problem 6.5. *Find $Z(D_n)$ and D_n'.*

Problem 6.6. *Draw a Cayley diagram for the group D_4 with the generators of your choice.*

Definition 6.7. The quaternion group Q_8 is defined by

$$Q_8 := \langle \varepsilon, i, j \mid \varepsilon^2 = \varepsilon i^2 = \varepsilon j^2 = \varepsilon i \varepsilon i = \varepsilon j \varepsilon j = \varepsilon i j i j = e \rangle. \tag{6.3}$$

Problem 6.8. *Show that*

$$|Q_8| = 8. \tag{6.4}$$

Problem 6.9. *Are the groups D_4 and Q_8 isomorphic?*

Problem 6.10. *Let X be a set of n elements. Let F_n denote the free group $F(X)$. Show that*

$$(F_n \cong F_m) \iff (n = m). \tag{6.5}$$

Problem 6.11. *Suppose that G, H are groups and $\alpha \in \mathrm{Hom}(G, H)$. Is it always true that*

$$\alpha(G') \leq H'? \tag{6.6}$$

Problem 6.12. *Define*

$$\mathrm{GL}_n(\mathbb{Z}) := \{A \in \mathrm{Mat}(n \times n, \mathbb{Z}) \mid \det(A) = 1 \text{ or } \det(A) = -1\}. \tag{6.7}$$

Using abelianization, construct a natural homomorphism from the group $\mathrm{Aut}(F_n)$ to the group $\mathrm{GL}_n(\mathbb{Z})$.

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