INTRODUCTORY NOTES 1. MAPS OF SETS

Please treat every proposition in this text as an exercise.

1. Composition of maps.

Definition 1.1. Let X and Y be sets. The Cartesian product $X \times Y$ is a set that consists of all possible pairs (x, y), where $x \in X, y \in Y$.

Exercise 1.2. Suppose that A, B, C and D are non-empty sets. Is it true that

$$(A \times B \subset C \times D) \iff ((A \subset C) \wedge (B \subset D))? \tag{1.1}$$

Definition 1.3. Let X, Y be sets, and let $P(x, y)$ be a property pertaining to an element $x \in X$ and an element $y \in Y$, such that for every $x \in X$, there is exactly one $y \in Y$ for which $P(x, y)$ is true. Then we define the map $f : X \to Y$ to be the object which, given any input $x \in X$, assigns an output $y \in Y$ for which $P(x, y)$ is true. We denote this output by $f(x)$. The set X is called the domain of f, the set Y is called the codomain of f.

Note that the domain may be the empty set.

Definition 1.4. Two maps $f : X \to Z$, $g : Y \to Z$ are said to be equal, written $f = g$, if and only if $X = Y$ and $f(x) = g(x)$ for all $x \in X$.

If two maps differ only by their codomains then, formally speaking, it would be natural to call them not equal. However, this situation does not arise in practice, because we usually have some freedom in changing the codomain; therefore the previous definition does not address it.

Definition 1.5. Let $f : X \to Y$ and $g : Z \to T$ be two maps, such that $Y \subset Z$. We then define the composition $g \circ f : X \to T$ of g and f to be the map defined explicitly by the formula $(g \circ f)(x) := g(f(x))$.

Proposition 1.6. *Composition is associative, namely, let $f : Z \to W$, $g : Y \to Z$, and $h : X \to Y$ be maps. Then $f \circ (g \circ h) = (f \circ g) \circ h$.*

Exercise 1.7. Do there exist maps $f, g : X \to X$ such that

$$f \circ g \neq g \circ f? \tag{1.2}$$

Exercise 1.8. Let X, Y and Z be sets and let $p_X : X \times Y \to X, p_Y : X \times Y \to Y$ be defined by

$$p_X((x, y)) := x; \quad p_Y((x, y)) := y, \quad \forall x \in X, \forall y \in Y. \tag{1.3}$$

Is it true that for every pair of maps $f : Z \to X$ and $g : Z \to Y$ there exists a unique map $h : Z \to X \times Y$ such that

$$p_X \circ h = f \text{ and } p_Y \circ h = g? \tag{1.4}$$

2. Injectivity, Surjectivity.

Definition 2.1. A map $f : X \to Y$ is called injective if $\forall x_1, x_2 \in X$

$$(f(x_1) = f(x_2)) \Longrightarrow (x_1 = x_2). \tag{2.1}$$

Definition 2.2. A map $f : X \to Y$ is called surjective if $\forall y \in Y \, \exists x \in X$ such that $f(x) = y$.

Definition 2.3. A map $f : X \to Y$ is called bijective if it is both injective and surjective.

Exercise 2.4. Give an example of a map that is
a) surjective but not injective;
b) injective but not surjective;
c) neither injective nor surjective.

Exercise 2.5. a) Does the composition of injective maps have to be injective?
b) Does the composition of surjective maps have to be surjective?
c) Does the composition of bijective maps have to be bijective?

Definition 2.6. Suppose $f : X \to Y$ is a map and $Z \subset X$. The restriction of the map f to the set Z is, by definition, a map

$$f|_Z : Z \to Y \tag{2.2}$$

defined by

$$f|_Z(z) := f(z), \quad \forall z \in Z. \tag{2.3}$$

Exercise 2.7. Show that for every map $f : X \to Y$ there exists a subset Z of the domain X such that $f|_Z$ is injective and

$$f|_Z(Z) = f(X). \tag{2.4}$$

What is an analog of this claim if we replace injectivity by surjectivity?

Exercise 2.8. Suppose that $f, g : X \to Y, h : Y \to Z$ are maps and

$$h \circ f = h \circ g. \tag{2.5}$$

Which condition on h (bijectivity, injectivity, surjectivity, no condition required) would allow us to claim $f = g$?

Exercise 2.9. Suppose that $f : X \to Y$, $g : Y \to Z$ are maps and the map $g \circ f$ is injective. Do both maps f and g have to be injective? What will change if we replace the condition of injectivity with surjectivity?

3. Preimage of a subset

Definition 3.1. Suppose $f : X \to Y$ is a map, $Z \subset X$, and $W \subset Y$. A set

$$f(Z) := \{f(z) \mid z \in Z\} \tag{3.1}$$

is called the image of the set Z. A set

$$f^{-1}(W) := \{x \in X \mid f(x) \in W\} \tag{3.2}$$

is called the preimage (the inverse image) of the set W.

Exercise 3.2. Suppose that $f : X \to Y$ is a map, $A \subset X$, and $B \subset Y$.
a) What can we say about the relation between the sets

$$f(f^{-1}(B)) \text{ and } B? \tag{3.3}$$

Is one a subset of another? Are they equal?
b) What can we say about the relation between the sets

$$f^{-1}(f(A)) \text{ and } A? \tag{3.4}$$

Exercise 3.3. Suppose that $f : X \to Y$ is a map, and $A, B \subset X$.
a) What can we say about the relation between the sets

$$f(A \cup B) \text{ and } f(A) \cup f(B)? \tag{3.5}$$

b) What can we say about the relation between the sets

$$f(A \cap B) \text{ and } f(A) \cap f(B)? \tag{3.6}$$

c) What can we say about the relation between the sets

$$f(A \setminus B) \text{ and } f(A) \setminus f(B)? \tag{3.7}$$

Exercise 3.4. Suppose that $f : X \to Y$ is a map, and $A, B \subset Y$.
a) What can we say about the relation between the sets

$$f^{-1}(A \cup B) \text{ and } f^{-1}(A) \cup f^{-1}(B)? \tag{3.8}$$

b) What can we say about the relation between the sets

$$f^{-1}(A \cap B) \text{ and } f^{-1}(A) \cap f^{-1}(B)? \tag{3.9}$$

c) What can we say about the relation between the sets

$$f^{-1}(A \setminus B) \text{ and } f^{-1}(A) \setminus f^{-1}(B)? \tag{3.10}$$

Exercise 3.5. Reformulate Definitions 2.1, 2.2 and 2.3 in terms of the images and preimages of sets.

Definition 3.6. Let X and Y be sets. Denote by Y^X the set of all maps from X to Y. Denote by 2^X the set of all subsets of X.

4. Inverse map

Exercise 4.1. Let $|X| = n$, $|Y| = m$. Find $|Y^X|$ and $|2^X|$. How many injective maps from X to Y are there?

Definition 4.2. For every set X define the identity map $\mathrm{Id}_X : X \to X$ by

$$\mathrm{Id}_X(x) := x, \quad \forall x \in X. \tag{4.1}$$

Exercise 4.3. Suppose that the maps $f, g : X \to X$ satisfy the equality

$$g \circ f = \mathrm{Id}_X. \tag{4.2}$$

Must the map f be injective or surjective? Must it be bijective?

Definition 4.4. Let $f : X \to Y$ be a bijection. Define the inverse map $f^{-1} : Y \to X$ by the following rule. Consider an arbitrary $y \in Y$. By our assumption $\exists! x \in X$ such that $f(x) = y$; then let

$$f^{-1}(y) := x. \tag{4.3}$$

Note the potentially confusing coincidence of notation between the inverse map and the preimage.

Exercise 4.5. Suppose that $f : X \to Y$ and $g : Y \to Z$ are bijections. Is it always true that:
a)

$$\left(f^{-1}\right)^{-1} = f? \tag{4.4}$$

b)

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}? \tag{4.5}$$