LECTURE 2. BASIS, DIMENSION

Throughout this lecture, we will assume that $\mathbb{F}$ is a field and V is a vector space over $\mathbb{F}$. Please consider every proposition in this text as an exercise.

1. Linear Span and Linear Independence

Definition 1.1. For a list of vectors $(v_1, \dots, v_n)$ we define its linear span to be the set

$$\boxed{\langle v_1, \dots, v_n \rangle := \{\sum_{j=1}^{n} a_j v_j \mid a_1, \dots, a_n \in \mathbb{F}\}.} \tag{1.1}$$

The expression $\sum_{j=1}^{n} a_j v_j$ is called a linear combination of the vectors $v_1, \dots, v_n$. We consider a list of vectors instead of a set to allow a vector to appear in that list more than once.

We allow a list to be empty. In that case,

$$\langle \emptyset \rangle = \{0\} \tag{1.2}$$

Exercise 1.2. Show that $\langle v_1, \dots, v_n \rangle$ is a subspace of V.

Definition 1.3. An ordered list of vectors $(v_1, \dots, v_n)$ is called linearly dependent if $\exists (a_1, \dots, a_n) \in \mathbb{F}^n$ such that:

(1) $(a_1, \dots, a_n) \neq (0, \dots, 0)$;
(2)

$$\sum_{j=1}^{n} a_j v_j = 0. \tag{1.3}$$

If a list of vectors $(v_1, \dots, v_n)$ is not linearly dependent, it is called linearly independent.

A linearly independent list can be viewed as a set because it cannot contain repetitions of elements. A subset of a linearly independent set is also linearly independent.

Proposition 1.4. *A list of vectors $(u_1, \dots, u_m)$ is linearly dependent iff $\exists j$, $1 \leq j \leq m$ such that*

$$u_j \in \langle u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_m \rangle, \tag{1.4}$$

namely, $\exists a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_m \in \mathbb{F}^{(m-1)}$ such that

$$u_j = \sum_{i \neq j} a_i u_i. \tag{1.5}$$

Proposition 1.5 (Linear Dependence Lemma)**.** *Suppose that a list of vectors $(u_1, \dots, u_m)$ is linearly dependent, and $u_1 \neq 0$. Show that $\exists j$, $2 \leq j \leq m$ such that the following hold:*

(1) $u_j \in \langle u_1, \dots, u_{j-1} \rangle$,
(2) *if the j-th term is removed from the list $(u_1, \dots, u_m)$ then the span of the remaining list equals $\langle u_1, \dots, u_m \rangle$:*

$$\langle u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_m \rangle = \langle u_1, \dots, u_m \rangle \tag{1.6}$$

Now, we can prove a crucial result that will allow us to introduce the notion of the dimension of a vector space later.

Proposition 1.6. ***(Replacement Theorem)*** *Suppose that*

$$\langle v_1, \dots, v_n \rangle = V, \tag{1.7}$$

and a subset $\{u_1, \dots, u_m\}$ *of* V *is linearly independent. Then* $m \le n$.

2. A Basis of a Vector Space

Definition 2.1. A set of vectors $\{v_1, \dots, v_n\}$ is called a basis of a vector space V if the following two conditions hold:

(1) The set $\{v_1, \dots, v_n\}$ is linearly independent;
(2) $\langle v_1, \dots, v_n \rangle = V$.

Let us consider other equivalent characterizations of a basis.

Proposition 2.2. *Let* $\{v_1, \dots, v_n\}$ *be a set of vectors in* V. *The following conditions are equivalent:*

(1) $\{v_1, \dots, v_n\}$ *is a basis of* V.

(2) $\forall x \in V \; \exists! (a_1, \dots, a_n) \in \mathbb{F}^n$ *such that*

$$x = \sum_{j=1}^{n} a_j v_j. \tag{2.1}$$

(3)

$$V = \bigoplus_{j=1}^{n} \langle v_j \rangle. \tag{2.2}$$

(4) $\{v_1, \dots, v_n\}$ *is a maximal linearly independent set, that is,* $\{v_1, \dots, v_n\}$ *is linearly independent, but after adding any vector to this set, it becomes linearly dependent.*

(5) $\{v_1, \dots, v_n\}$ *is a minimal spanning set of* V, *that is,* $\{v_1, \dots, v_n\}$ *spans* V *but any proper subset of* $\{v_1, \dots, v_n\}$ *does not span* V.

Let us list corollaries of Propositions 1.6 and 2.2.

Proposition 2.3. *Let* $\{v_1, \dots, v_n\}$ *be a basis of* V. *Then*

(1) *Every list of vectors from* V *of length greater than* n *is linearly dependent;*

(2) *If* $\langle u_1, \dots, u_m \rangle = V$ *then* $m \ge n$;

(3) *If* $\langle w_1, \dots, w_n \rangle = V$ *then the set* $\{w_1, \dots, w_n\}$ *is a basis of* V;

(4) *If a set* $\{x_1, \dots, x_n\}$ *is linearly independent, then it is a basis of the vector space* V;

(5) *If a set* $\{y_1, \dots, y_m\}$ *is a basis of* V *then* $m = n$.

3. Dimension of a Vector Space and Its Properties

Definition 3.1. Let $\{v_1, \ldots, v_n\}$ be a basis of V. Then we say that V is a finite-dimensional space, and we write $\dim V < \infty$. The number n is called the dimension of the vector space V. We then write $\dim V = n$.

Note that $\dim\{0\} = 0$.

Proposition 3.2. *Let U be a proper subspace of a finite-dimensional vector space V. Show that U is also finite-dimensional and $\dim U < \dim V$.*

Next, we discuss how we can construct a basis from existing subsets of vectors (Propositions 3.3 and 3.6).

Proposition 3.3. *Suppose that $\dim V = n$ and $\{u_1, \ldots, u_m\}$ is a linearly independent subset of V, then there exist vectors $w_1, \ldots, w_{n-m}$ such that $\{u_1, \ldots, u_m, w_1, \ldots, w_{n-m}\}$ is a basis of V.*

Proposition 3.4. *Let U be a subspace of a finite-dimensional vector space V. Then there exists a subspace W such that*

$$V = U \oplus W. \tag{3.1}$$

Proposition 3.5. *Let U be a subspace of a finite-dimensional vector space V. Then the quotient space V/U is also finite-dimensional and*

$$\dim(V/U) = \dim V - \dim U. \tag{3.2}$$

Proposition 3.6. *Suppose that $\dim V = n$ and*

$$\langle u_1, \ldots, u_m \rangle = V. \tag{3.3}$$

Show that there exists a subset of $\{u_1, \ldots, u_m\}$ that is a basis of V.

Proposition 3.7. *Suppose that U and W are subspaces of a finite-dimensional vector space V. Then*

$$\boxed{\dim(W + U) = \dim W + \dim U - \dim(W \cap U).} \tag{3.4}$$

Exercise 3.8. Suppose that $\dim V < \infty$ and

$$V = \bigoplus_{j=1}^{k} U_j. \tag{3.5}$$

Then

$$\dim V = \sum_{j=1}^{k} \dim U_j. \tag{3.6}$$

Chapters from the textbooks relevant to the lecture:

- Friedberg, Insel, Spence, Sections 1.4 - 1.7;
- Roman, pp.44 - 51;
- Axler, ch. 2.