LECTURE 3. LINEAR MAPS

Throughout this lecture, we will assume that $\mathbb{F}$ is a field, and all vector spaces are defined over that same field. Please treat every proposition in this lecture as an exercise.

1. Types of Linear Maps

Definition 1.1. Suppose that V and U are vector spaces over the same field $\mathbb{F}$. A map $A : V \to U$ is called a linear map (or a linear transformation, or a homomorphism) if

$$\boxed{A(x+y) = Ax + Ay, \quad A(\lambda x) = \lambda Ax, \quad \forall x, y \in V, \forall \lambda \in \mathbb{F}.} \tag{1.1}$$

The set of all linear maps from V to U is denoted by $\mathrm{Hom}(V, U)$.

Definition 1.2. We define pointwise addition and multiplication by scalars in $\mathrm{Hom}(V, U)$:

$$(A+B)(x) := Ax + Bx, \quad (\lambda A)(x) := \lambda(Ax), \quad \forall A, B \in \mathrm{Hom}(V, U), \forall x \in V, \forall \lambda \in \mathbb{F}. \tag{1.2}$$

Exercise 1.3. Show that if $A, B \in \mathrm{Hom}(V, U)$ then $A + B, \lambda A \in \mathrm{Hom}(V, U)$.

Exercise 1.4. Show that $\mathrm{Hom}(V, U)$ is a vector space with respect to those operations.

Definition 1.5. We use the following terms and notation:

(1) A linear map from a vector space to itself is called an endomorphism or a linear operator. The set of all linear operators on V is denoted by $\mathrm{End}(V)$. The map $\mathrm{Id} \in \mathrm{End}(V)$ defined by
$$\mathrm{Id}(x) := x, \quad \forall x \in V \tag{1.3}$$
is called the identity map.
(2) A linear map is called a monomorphism or an embedding if it is injective.
(3) A linear map is called an epimorphism if it is surjective.
(4) A linear map is called an isomorphism if it is bijective. The set of all isomorphisms from a vector space V to a vector space U is denoted by $\mathrm{Iso}(V, U)$. If there exists an isomorphism from V to U, we will write $V \cong U$.
(5) A linear operator is called an automorphism if it is bijective. The set of all automorphisms of a vector space V is denoted by $\mathrm{GL}(V)$ (which stands for "General Linear (Group)") or by $\mathrm{Aut}(V)$.

A part of this notation is summarized in the following table.

	$V \to U$	$V \to V$
General case	$\mathrm{Hom}(V, U)$, homomorphisms	$\mathrm{End}(V)$, endomorphisms=linear operators
The map is invertible	$\mathrm{Iso}(V, U)$, isomorphisms	$\mathrm{Aut}(V) = \mathrm{GL}(V)$, automorphisms=invertible linear operators

Exercise 1.6. Which of the sets $\mathrm{End}(V)$, $\mathrm{Iso}(V, U)$, $\mathrm{GL}(V)$ are vector spaces with respect to the pointwise addition and the pointwise multiplication by scalars? Is the set of monomorphisms a subspace of $\mathrm{Hom}(V, U)$? Is the set of epimorphisms a subspace of $\mathrm{Hom}(V, U)$?

Definition 1.7. Suppose that V, U, W are finite-dimensional spaces, $A \in \mathrm{Hom}(V, U)$, $B \in \mathrm{Hom}(U, W)$. We also consider the operation of composition of linear maps B and A. We denote this composition of linear maps by BA:

$$BA(x) := B(A(x)), \quad \forall x \in V. \tag{1.4}$$

We use this notation because composition of linear operators shares several nice properties with multiplication.

Exercise 1.8. Is $\mathrm{End}(V)$ a group with respect to this operation? Is $\mathrm{GL}(V)$ a group with respect to the operation of composition? Is it always true that

$$A(B + C) = AB + AC, \quad \forall A, B, C \in \mathrm{End}(V)? \tag{1.5}$$

2. Image and Kernel of a Linear Map

Definition 2.1. For an arbitrary $A \in \mathrm{Hom}(V, U)$ we define

$$\mathrm{Ker}(A) := \{x \in V \mid Ax = 0\}, \quad \mathrm{Im}(A) := \{y \in U \mid \exists x \in V \text{ such that } Ax = y\}. \tag{2.1}$$

The set $\mathrm{Ker}(A)$ is called the kernel of A. The set $\mathrm{Im}(A)$ is called the image of A.

Proposition 2.2. *For an arbitrary $A \in \mathrm{Hom}(V, U)$ the set $\mathrm{Ker}(A)$ is a subspace of the vector space V, and $\mathrm{Im}(A)$ is a subspace of the vector space U.*

Exercise 2.3. Show that

$$(A \text{ is a monomorphism}) \iff (\mathrm{Ker}A = \{0\}). \tag{2.2}$$

Next, we will introduce a version of the fundamental theorem on homomorphisms for vector spaces.

Proposition 2.4. *(The First Isomorphism Theorem) For an arbitrary $A \in \mathrm{Hom}(V, U)$ define the map $\bar{A}$ from the quotient space $V/\mathrm{Ker}(A)$ to the vector space U by*

$$\bar{A}(x + \mathrm{Ker}(A)) := Ax, \quad \forall x \in V. \tag{2.3}$$

Then, $\bar{A}$ is a well-defined linear map. Moreover, $\bar{A}$ is an isomorphism between the quotient space $V/\mathrm{Ker}(A)$ and $\mathrm{Im}(A)$.

Definition 2.5. Suppose that V and U are finite-dimensional vector spaces. For a linear map $A \in \mathrm{Hom}(V, U)$ we define

$$\mathrm{rk}(A) := \dim(\mathrm{Im}(A)). \tag{2.4}$$

The number $\mathrm{rk}(A)$ is called the rank of the homomorphism A. The number $\dim(\mathrm{Ker}(A))$ is called the nullity of A.

Proposition 2.6. *(The Rank Plus Nullity Theorem) Suppose that V and U are finite-dimensional vector spaces. For a linear map $A \in \mathrm{Hom}(V, U)$ we get the identity*

$$\boxed{\mathrm{rk}(A) + \dim(\mathrm{Ker}(A)) = \dim V.} \tag{2.5}$$

3. Isomorphisms

The following two propositions will provide another way to look at isomorphisms.

Proposition 3.1. *A linear map $A \in \mathrm{Hom}(V,U)$ is an isomorphism if and only if* $\mathrm{Ker}(A) = \{0\}$ *and* $\mathrm{Im}(A) = U$. *Then the inverse map A^{-1} is also linear and, therefore, $A^{-1} \in \mathrm{Iso}(U,V)$.*

Proposition 3.2. *Suppose that V and U are vector spaces and $A \in \mathrm{Hom}(V,U)$. Then*

$$(A \in \mathrm{Iso}(V,U)) \quad \Longleftrightarrow \quad (\exists B \in \mathrm{Hom}(U,V) \text{ such that } AB = BA = \mathrm{Id}). \tag{3.1}$$

If we can pick a finite basis in a vector space, then we may construct an isomorphism of that space with one of the standard vector spaces $\mathbb{F}^n$.

Proposition 3.3. *Suppose that V is a finite-dimensional space, $\alpha = (v_1, \dots, v_n)$ is a basis of V. We assume that the basis is ordered, i.e., the order of the elements in the set matters. We use parentheses instead of curly brackets to emphasize that we consider an ordered sequence.*

As we know, $\forall x \in V \; \exists!(a_1, \dots, a_n) \in \mathbb{F}^n$ such that

$$x = \sum_{j=1}^{n} a_j v_j. \tag{3.2}$$

The numbers $(a_1, \dots, a_n)$ are called the coordinates of the vector x in the basis α. Let us define a map $\varphi_\alpha : V \to \mathbb{F}^n$ by setting

$$\varphi_\alpha(x) := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}. \tag{3.3}$$

Then the map $\varphi_\alpha \in \mathrm{Iso}(V, \mathbb{F}^n)$.

That claim allows us to provide the following classification of finite-dimensional vector spaces over the same field up to an isomorphism.

Proposition 3.4. *Suppose that V and U are finite-dimensional vector spaces. There exists an isomorphism between V and U (or we may say that V and U are isomorphic) if and only if* $\dim V = \dim U$.

So, to find out whether two finite-dimensional vector spaces are isomorphic, it is sufficient to check only their dimensions. Consider the equivalence relation of being isomorphic on finite-dimensional vector spaces over a fixed field $\mathbb{F}$. Then, every equivalence class consists of all finite-dimensional vector spaces of the same dimension, including the standard space $\mathbb{F}^n$ for some n.

Note that to figure out whether a linear map between two vector spaces of the same finite dimension is an isomorphism, it is sufficient to check either injectivity or surjectivity.

Proposition 3.5. *Suppose that V and U are finite-dimensional vector spaces and* $\dim V = \dim U$.

a) If $A \in Hom(V,U)$ is a monomorphism then A is an isomorphism.

b) If $A \in Hom(V,U)$ is an epimorphism then A is an isomorphism.

Chapters from the textbooks relevant to the lecture:

- Friedberg, Insel, Spence, Sections 2.1 - 2.4;
- Roman, pp.59 - 65;
- Axler, Sections 3A, 3B, 3D.