LECTURE 4. MATRIX OF A LINEAR MAP

Throughout this lecture, we will assume that $\mathbb{F}$ is a field, and vector spaces are defined over that same field. Please treat every proposition in this lecture as an exercise.

1. The Matrix of a Linear Map

Proposition 1.1. *Suppose that $(v_1, \dots, v_n)$ is a basis of a vector space V.*

a) For every list of vectors $y_1, \dots, y_n \in U$ there exists a unique map $B \in \mathrm{Hom}(V, U)$ such that $Bv_j = y_j$, $\forall 1 \le j \le n$.

b) Every map $A \in \mathrm{Hom}(V, U)$ is uniquely determined by the ordered set of images $Av_1, \dots, Av_n$.

The next definition establishes how we can represent linear maps via matrices.

Definition 1.2. Suppose that $\alpha = (v_1, \dots, v_n)$ is an ordered basis of a vector space V, $\beta = (u_1, \dots, u_m)$ is an ordered basis of a vector space U, $A \in \mathrm{Hom}(V, U)$. Then there exists a list of numbers (a_i^j) such that

$$\boxed{Av_i = \sum_{j=1}^{m} a_i^j u_j, \quad \forall 1 \le i \le n.} \tag{1.1}$$

The matrix filled with entries (a_i^j)

$$\begin{pmatrix} a_1^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \dots & a_n^m \end{pmatrix} \tag{1.2}$$

is called the matrix of the map A (in the bases α and β) and is denoted by the same letter A or by $[A]_\alpha^\beta$. The set of $m \times n$ matrices with entries from the field $\mathbb{F}$ is denoted by $\mathrm{Mat}(m \times n, \mathbb{F})$.

Exercise 1.3. Suppose that V and U are finite-dimensional vector spaces with a fixed ordered basis in each of them, $A, B \in \mathrm{Hom}(V, U)$. Express the entries of the matrices of the linear maps $A + B$ and λA using the entries of A and B.

Exercise 1.4. It is obvious that the set $\mathrm{Mat}(m \times n, \mathbb{F})$ is a vector space with respect to entrywise addition and entrywise multiplication by scalars. Suppose that V and U are finite-dimensional vector spaces. Show that the vector spaces $\mathrm{Hom}(V, U)$ and $\mathrm{Mat}(\dim U \times \dim V, \mathbb{F})$ are isomorphic.

Exercise 1.5. Suppose that V and U are finite-dimensional vector spaces. Show that

$$\boxed{\dim(\mathrm{Hom}(V, U)) = \dim V \cdot \dim U.} \tag{1.3}$$

Now, we will establish the rule of multiplication of matrices that corresponds to the composition of linear maps.

Definition 1.6. (The Row-Column rule) Suppose that V, U, W are finite-dimensional spaces with bases α, β, γ respectively, $A \in \mathrm{Hom}(V, U), B \in \mathrm{Hom}(U, W)$. We define the product of matrices via the formula

$$\boxed{[B]_\beta^\gamma \cdot [A]_\alpha^\beta := [BA]_\alpha^\gamma.} \tag{1.4}$$

Exercise 1.7. Suppose that X is an arbitrary $m \times n$ matrix, Y is an arbitrary $p \times m$ matrix. Show that Definition 1.6 defines a unique product $Y \cdot X$, which is a $p \times n$ matrix. Find a formula expressing the entries of the matrix $[BA]_\alpha^\gamma$ using the entries of the matrices $[A]_\alpha^\beta$ and $[B]_\beta^\gamma$.

Using the Row-Column rule notation, we may rewrite the Definition 1.2 as the following formal expression

$$(Av_1, \ldots, Av_n) = (u_1, \ldots, u_m) \begin{pmatrix} a_1^1 & \ldots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \ldots & a_n^m \end{pmatrix}. \tag{1.5}$$

2. Change of Bases for Linear Maps

Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v_1', \ldots, v_n')$ are two ordered bases of a vector space V. Recall that in the previous lecture, we introduced the notation $\varphi_\alpha(x)$ for the linear map from V to $\mathbb{F}^n$ assigning to a vector x its coordinates in basis α. Also, we used the notation $[A]_\alpha^\beta$ for the matrix of a linear map A expressing the images of the vectors of the basis α as linear combinations of the vectors of the basis β. In our current notation, it takes the following form:

$$Av_i = \sum_{j=1}^{n} a_i^j v_j', \quad \forall 1 \le i \le n, \quad [A]_\alpha^\beta = (a_i^j)_{1 \le i,j \le n}. \tag{2.1}$$

Before defining the change-of-basis matrix, we will establish some of its properties.

Proposition 2.1. *Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v_1', \ldots, v_n')$ are two ordered bases of a vector space V.*

a) Then the matrix $[\mathrm{Id}]_\alpha^\beta$ is invertible.

b) The matrix $[\mathrm{Id}]_\alpha^\beta$ is equal to the matrix of the linear operator $\varphi_\beta \circ (\varphi_\alpha)^{-1}$ in the standard basis of $\mathbb{F}^n$.

Definition 2.2. Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v_1', \ldots, v_n')$ are two ordered bases of a vector space V. The matrix $[\mathrm{Id}]_\alpha^\beta$ is called the change of basis matrix from α to β.

From Proposition 2.1, we see that the coordinates of a vector in one basis can be obtained from its coordinates in the other by applying the change-of-basis matrix:

$$\varphi_\beta(x) = [\mathrm{Id}]_\alpha^\beta \varphi_\alpha(x). \tag{2.2}$$

Recall the following rule connecting the composition of linear maps with matrix multiplication.

$$[B]_\beta^\gamma \cdot [A]_\alpha^\beta := [BA]_\alpha^\gamma. \tag{2.3}$$

We use it to establish the following claim.

Proposition 2.3. *Suppose that $\alpha = (v_1, \ldots, v_n)$ and $\beta = (v_1', \ldots, v_n')$ are two ordered bases of a vector space V, $\gamma = (u_1, \ldots, u_m)$, and $\delta = (u_1', \ldots, u_m')$ are two ordered bases of a vector space U, C_1 is the change of basis matrix from α to β, C_2 is the change of basis matrix from γ to δ, and $A \in \mathrm{Hom}(V, U)$. Then the matrices $[A]_\alpha^\gamma$ and $[A]_\beta^\delta$ satisfy the following relation:*

$$\boxed{[A]_\beta^\delta = C_2 \, [A]_\alpha^\gamma \, C_1^{-1}.} \tag{2.4}$$

Let us consider the case of linear operators. We agree that for an operator $A \in \mathrm{End}(V)$, we pick a single basis α in V. So the matrices of endomorphisms A have the form $[A]_\alpha^\alpha$ for some basis α. Then, we may reformulate Proposition 2.3 for that case.

Proposition 2.4. *Suppose that $\alpha = (v_1, \dots, v_n)$ and $\beta = (v'_1, \dots, v'_n)$ are two ordered bases of a vector space V, C is the change of basis matrix from α to β, and $A \in \mathrm{End}(V)$. Then the matrices $[A]^\alpha_\alpha$ and $[A]^\beta_\beta$ satisfy the following relation:*

$$\boxed{[A]^\beta_\beta = C[A]^\alpha_\alpha C^{-1}.} \tag{2.5}$$

3. Equivalence of Matrices

Following the result of Proposition 2.3, we define the equivalence of matrices.

Definition 3.1. Suppose that $A, B \in \mathrm{Mat}(m \times n, \mathbb{F})$. They are called equivalent if there exist invertible matrices $C \in \mathrm{Mat}(m \times m, \mathbb{F})$, $D \in \mathrm{Mat}(n \times n, \mathbb{F})$ such that

$$A = C\,B\,D^{-1}. \tag{3.1}$$

The analogous notion for linear operators is called similarity of square matrices.

Definition 3.2. Suppose that $A, B \in \mathrm{Mat}(n \times n, \mathbb{F})$. They are called similar if there exists an invertible matrix $C \in \mathrm{Mat}(n \times n, \mathbb{F})$ such that

$$A = C\,B\,C^{-1}. \tag{3.2}$$

Using the ideas behind the First Isomorphism Theorem, we can essentially solve the problem of classification of equivalent matrices.

Proposition 3.3. *Suppose that V and U are finite-dimensional vector spaces, $A \in \mathrm{Hom}(V, U)$, and $\mathrm{rk}(A) = k$. Then there exist bases α and β of the spaces V and U, respectively, such that the matrix of the linear map has the following block diagonal form:*

$$\boxed{[A]^\beta_\alpha = \begin{pmatrix} \mathrm{Id}_k & 0 \\ 0 & 0 \end{pmatrix},} \tag{3.3}$$

where Id_k is the $k \times k$ identity matrix.

Therefore, we see that every equivalence class of matrices of the same size consists of the matrices of the same rank.

The problem of classification of similar matrices turns out to be more complicated. We are going to deal with it later in the course.

4. System of Linear Equations

A system of linear equations

$$\sum_{i=1}^{n} a^j_i x_i = b_j, \quad 1 \le j \le m \tag{4.1}$$

can be represented as $Ax = b$, where

$$A = (a^j_i)_{1 \le i \le n,\, 1 \le j \le m}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}. \tag{4.2}$$

This notation allows us to formulate two general statements about those systems. We are going to view A both as a matrix and as a linear map from $\mathbb{F}^n$ to $\mathbb{F}^m$ defined by that matrix in the standard bases.

Proposition 4.1. *Solutions of the system $Ax = 0$ form a subspace in the vector space $\mathbb{F}^n$. The dimension of the subspace of solutions of the system $Ax = 0$ equals $n - \mathrm{rk}(A)$. In particular, if $m < n$, then there exists a non-zero solution to the homogeneous system.*

Proposition 4.2. *A solution of the system $Ax = b$ exists if and only if $b \in \mathrm{im}A$. In particular, if $\mathrm{rk}(A) = m$, then a solution exists. The difference between any two solutions of the system $Ax = b$ is itself a solution of the system $Ax = 0$.*

Chapters from the textbooks relevant to the lecture:

- Friedberg, Insel, Spence, Sections 2.2-2.5;
- Roman, pp.65 - 72;
- Axler, Sections 3C, 3D.