LECTURE 6. SYMMETRIC AND EXTERIOR POWERS, DETERMINANT

Throughout this lecture, we will assume that $\mathbb{F}$ is a field of characteristic 0, and vector spaces are defined over that same field. Please consider the proof of every proposition in this lecture to be an exercise.

1. Symmetric Group

Definition 1.1. A map $\rho : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is called a permutation (of the set $\{1, 2, \dots, n\}$) if ρ is a bijection. The set of all permutations of the set $\{1, 2, \dots, n\}$ is called the symmetric group and is denoted by S_n. The composition $\rho \circ \sigma$ of two permutations $\rho, \sigma \in S_n$ is denoted by $\rho\sigma$. For an element $\rho \in S_n$ we may use a two-row notation:

$$\rho = \begin{pmatrix} 1 & 2 & \dots & n \\ \rho(1) & \rho(2) & \dots & \rho(n) \end{pmatrix}. \tag{1.1}$$

A more common notation is cycle notation. It is defined in the following way. Let us fix an arbitrary permutation $\rho \in S_n$. We introduce an equivalence relation on the set $\{1, 2, \dots, n\}$ by the following rule:

$$(i \sim j) \quad \Longleftrightarrow \quad (\exists k \in \mathbb{Z}_+ \text{ such that } \rho^k(i) = j), \tag{1.2}$$

where

$$\rho^k := \underbrace{\rho \circ \rho \circ \dots \circ \rho}_{k \text{ times}}. \tag{1.3}$$

One may check that it is indeed an equivalence relation. For each equivalence class that consists of, say, m elements, we pick a representative j and write a cycle $(j, \rho(j), \rho^2(j), \dots, \rho^{m-1}(j))$ in the cycle notation of ρ. If $m = 1$, we usually omit the corresponding one-element cycle. Here is an example of cycle notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 1 & 4 & 3 \end{pmatrix} = (1, 5, 4)(3, 6). \tag{1.4}$$

One may show that cycle notation is well-defined. A cycle of length 2 is called a transposition.

Here is an example of the product of two permutations written in cycle notation:

$$((1, 2)(3, 4, 5))((2, 3)(1, 5)) = (1, 3)(2, 4, 5). \tag{1.5}$$

Exercise 1.2. Show that S_n is a group with respect to composition and find its order.

One may compare the following statement with the change of basis for linear operators.

Exercise 1.3. Show that for every permutation $\rho \in S_n$ and for every subset $\{i_1, i_2, \dots, i_k\}$ of the set $\{1, 2, \dots, n\}$ the following equality holds

$$\boxed{\rho \circ (i_1, i_2, \dots, i_k) \circ \rho^{-1} = (\rho(i_1), \rho(i_2), \dots, \rho(i_k)),} \tag{1.6}$$

where the LHS is the composition of three permutations such that the second permutation is a cycle of length k.

Exercise 1.4. Suppose that $\rho \in S_n$. Consider a map $\psi_\rho : S_n \to S_n$ defined by

$$\psi_\rho(\sigma) := \rho\sigma\rho^{-1}. \tag{1.7}$$

Show that ψ_ρ is an automorphism of the group S_n.

Exercise 1.5. Show that every permutation $\rho \in S_n$ can be represented as a product of some transpositions.

Proposition 1.6. *For every positive integer n there exists a unique map* $\mathrm{sgn} : S_n \to \{1, -1\}$ *such that*

(1) $\mathrm{sgn}((i,j)) = -1, \quad \forall 1 \le i < j \le n;$
(2) $\mathrm{sgn}(\rho\sigma) = \mathrm{sgn}(\rho)\mathrm{sgn}(\sigma).$

The value $\mathrm{sgn}(\rho)$ *is called the sign of the permutation* ρ.

To prove this proposition, one may consider the ratio

$$\frac{\prod_{1\le i<j\le n}(x_{\rho(i)} - x_{\rho(j)})}{\prod_{1\le i<j\le n}(x_i - x_j)}. \tag{1.8}$$

2. Symmetric and Exterior Powers

Definition 2.1. Let V be a vector space over $\mathbb{F}$.

Define the k-th symmetric and the k-th exterior powers of V, denoted $S^k(V)$ and $\Lambda^k(V)$, respectively, as the following subspaces of $V^{\otimes k}$:

$$S^k(V) := \mathrm{span}\langle \sum_{\rho\in S_k} x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)} \,|\, x_1, x_2, \ldots, x_k \in V\rangle, \tag{2.1}$$

$$\Lambda^k(V) := \mathrm{span}\langle \sum_{\rho\in S_k} \mathrm{sgn}(\rho) x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)} \,|\, x_1, x_2, \ldots, x_k \in V\rangle. \tag{2.2}$$

We use the notation

$$x_1 \wedge x_2 \wedge \cdots \wedge x_k := \frac{1}{k!} \sum_{\rho\in S_k} \mathrm{sgn}(\rho) x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)}. \tag{2.3}$$

Exercise 2.2. Suppose that $\dim V = n$. Find $\dim S^k(V)$ and $\dim \Lambda^k(V)$.

Proposition 2.3. *Let* $A \in \mathrm{End}(V)$, *then*

$$\forall z \in S^k(V),\ A^{\otimes k}(z) \in S^k(V), \tag{2.4}$$

$$\forall z \in \Lambda^k(V),\ A^{\otimes k}(z) \in \Lambda^k(V). \tag{2.5}$$

Definition 2.4. Let $A \in \mathrm{End}(V)$. Denote the restrictions of $A^{\otimes k}$ to the subspaces $S^k(V)$ and $\Lambda^k(V)$ by $S^k(A) \in \mathrm{End}(S^k(V))$ and by $\Lambda^k(A) \in \mathrm{End}(\Lambda^k(V))$, respectively.

3. Determinant

Definition 3.1. Assume that V is a finite-dimensional vector space, $A \in \mathrm{End}(V)$. Note that $\dim(\Lambda^{\dim V}(V)) = 1$. Therefore, $\Lambda^{\dim V}(A)$ is a multiple of the identity map. We will define the determinant $\det(A)$ of the linear operator A by the following equality

$$\boxed{\Lambda^{\dim V}(A) = \det(A) \cdot \mathrm{Id}.} \tag{3.1}$$

Proposition 3.2. *Suppose that* $\dim V < \infty$ *and* $A, B \in \mathrm{End}(V)$. *Then*

$$\boxed{\det(AB) = \det(A) \cdot \det(B).} \tag{3.2}$$

Proposition 3.3. *Suppose that* $\dim V = n < \infty$ *and* $A \in \mathrm{End}(V)$ *has a matrix* $(a_i^j)_{1 \le i,j \le n}$ *in some basis. Then*

$$\boxed{\det(A) = \sum_{\rho \in S_n} \mathrm{sgn}(\rho) \prod_{j=1}^{n} a^j_{\rho(j)}.} \tag{3.3}$$

Chapters from the textbooks relevant to the lecture:

- Roman, pp. 363-378, 393-406,
- Axler, Section 9C.