LECTURE 1. VECTOR SPACES

1. Preliminary Algebraic Definitions

Definition 1.1. Consider a set G with a map mult : $G \times G \to G$. We are going to call such maps binary operations. In this case we will denote the image mult(a, b) by ab. The pair $(G, mult)$ is called a monoid if the following conditions hold:

(1) $a(bc) = (ab)c, \quad \forall a, b, c \in G$;
(2) $\exists e \in G$ such that $ae = ea = a, \quad \forall a \in G$ (e is called the identity element).

Exercise 1.2. Show that the identity element e in a monoid is unique.

Definition 1.3. A monoid G is called a group if $\forall a \in G\, \exists b \in G$ such that

$$ab = ba = e. \tag{1.1}$$

An element b is called the inverse element of a and is denoted by a^{-1}.

Exercise 1.4. Show that for every element of a group its inverse element is unique.

Definition 1.5. A group G such that

$$\forall a, b \in G\, ab = ba \tag{1.2}$$

is called abelian. In an abelian group the group operation is usually denoted by $+$ and the inverse element to an element a is denoted by $-a$.

Question 1.6. Does there exist a non-abelian group?

Definition 1.7. For two groups G and H a map $f : G \to H$ is called a homomorphism of groups if

$$f(ab) = f(a)f(b), \quad \forall a, b \in G. \tag{1.3}$$

If a homomorphism is also a bijection then it is called an isomorphism of the groups.

Note that the multiplications involved in the LHS and the RHS of the equality (1.3) are different. In the LHS we multiply a and b in the group G. In the RHS we multiply their images in the group H.

Exercise 1.8. For a homomorphism f of groups G an H show that:
a) $f(e_G) = e_H$;
b) $\forall a \in G\, (f(a))^{-1} = f(a^{-1})$.

Exercise 1.9. Show that being isomorphic is an equivalence relation on the set of groups.

Definition 1.10. A set $\mathbb{F}$ with operations of multiplication and addition is called a field if the following conditions hold:

(1) $(\mathbb{F}, +)$ is an abelian group. We denote its identity element by 0.
(2) $(\mathbb{F} \setminus \{0\}, \cdot)$ is an abelian group. We denote its identity element by 1.

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(3) $a(b+c) = ab + ac$, $(a+b)c = ac + bc$, $\quad \forall a, b, c \in \mathbb{F}$.

Exercise 1.11. For a field $\mathbb{F}$ show that $a \cdot 0 = 0$, $\forall a \in \mathbb{F}$.

Exercise 1.12. Which of the following sets are fields with respect to standard operations:

$$\emptyset, \quad \{0\}, \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{R}_+, \quad \mathbb{R}, \quad \mathbb{C} \quad ?$$

2. Definition of a Vector Space

Definition 2.1. Assume that $\mathbb{F}$ is a field, whose elements are referred to as scalars. A non-empty set V with an operation of addition and an operation of scalar multiplication scal : $\mathbb{F} \times V \to V$ $((\lambda, x) \mapsto \lambda x)$ is called a vector space over $\mathbb{F}$ if the following conditions hold:

(1) $(V, +)$ is an abelian group. We denote its identity element by 0 or by 0_V.
(2) $\lambda(x + y) = \lambda x + \lambda y, \quad \forall \lambda \in \mathbb{F}, \forall x, y \in V$.
(3) $(\lambda + \mu)x = \lambda x + \mu x, \quad \forall \lambda, \mu \in \mathbb{F}, \forall x \in V$.
(4) $(\lambda\mu)x = \lambda(\mu x), \quad \forall \lambda, \mu \in \mathbb{F}, \forall x \in V$.
(5) $1_{\mathbb{F}}\, x = x, \quad \forall x \in V$.

The notation here may be a bit confusing. For example, in condition (2) the symbol"+" in the LHS stands for the addition in the field $\mathbb{F}$, and in the RHS the same symbol stands for the addition in V.

Exercise 2.2. For a vector space V over a field $\mathbb{F}$ show that $0_{\mathbb{F}}\, x = 0_V$, $\forall x \in V$.

Exercise 2.3. For a vector space V over a field $\mathbb{F}$ show that $(-1_{\mathbb{F}})\, x = -x$, $\forall x \in V$.

Definition 2.4. For a field $\mathbb{F}$ denote by

$$\mathbb{F}^n := \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mid a_j \in \mathbb{F}, \forall 1 \le j \le n \right\} \tag{2.1}$$

the set of ordered n-tuples of elements of $\mathbb{F}$. We define the component-wise addition on $\mathbb{F}^n$ and the multiplication of elements of $\mathbb{F}^n$ by the scalars of $\mathbb{F}$:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} := \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}, \quad \forall a_j, b_j \in \mathbb{F}; \tag{2.2}$$

$$\lambda \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} := \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{pmatrix}, \quad \forall a_j, \lambda \in \mathbb{F}. \tag{2.3}$$

Exercise 2.5. Which of the following pairs are vector spaces over corresponding fields:

(1) $\emptyset$ over $\mathbb{R}$;
(2) $\{0\}$ over $\mathbb{R}$
(3) $\mathbb{Z}$ over $\mathbb{Q}$;
(4) $\mathbb{Q}$ over $\mathbb{Z}$;
(5) $\mathbb{Q}$ over $\mathbb{R}$;
(6) $\mathbb{R}$ over $\mathbb{Q}$;
(7) $\mathbb{C}$ over $\mathbb{R}$;
(8) $\mathbb{C}^n$ over $\mathbb{C}$;
(9) $\mathbb{C}^n$ over $\mathbb{R}$?

3. Subspaces, Direct Sums, Quotient Spaces

Definition 3.1. Assume that V is a vector space over a field $\mathbb{F}$. A non-empty subset U of V is called its subspace if:

(1) for every elements $x, y \in U$ their sum $x + y \in U$;
(2) for every scalar $\lambda \in \mathbb{F}$ and for every element $x \in U$ the product $\lambda x \in U$.

Definition 3.2. Assume that V is a vector space over a field $\mathbb{F}$. A non-empty subset U of V is called its subspace if U is a vector space over $\mathbb{F}$ with respect to addition and multiplication by scalars that are defined in V.

Exercise 3.3. Show that the previous two definitions are equivalent.

Note that it is widely agreed that if V is a vector space then the notation $U \subset V$ means that U is a subspace of V unless specified otherwise.

Exercise 3.4. Is it always true that the intersection of an arbitrary family of subspaces of a vector space V is also its subspace?

Definition 3.5. Assume that $U_1, \dots, U_n$ are subspaces of the vector space V. Define a set

$$\sum_{j=1}^{n} U_j := \{\sum_{j=1}^{n} x_j \mid x_j \in U_j, \forall 1 \le j \le n\}. \tag{3.1}$$

It is called the sum or the join of the subspaces $U_1, \dots, U_n$. Another notation for it is $U_1 + U_2 + \dots + U_n$.

Exercise 3.6. Is it always true that $\sum_{j=1}^{n} U_j$ is also a subspace of the vector space V?

Definition 3.7. Assume that $U_1, \dots, U_n$ are subspaces of the vector space V. We are going to write

$$V = \bigoplus_{j=1}^{n} U_j \tag{3.2}$$

and say that V is a direct sum of $U_1, \dots, U_n$ if and only if $\forall x \in V \; \exists!(x_1, \dots, x_n)$ such that

(1) $x_j \in U_j$, $\forall 1 \le j \le n$;
(2) $x = \sum_{j=1}^{n} x_j$.

Exercise 3.8. Show that $V = U_1 \oplus U_2$ if and only if

(1) $V = U_1 + U_2$;
(2) $U_1 \cap U_2 = \{0\}$.

Definition 3.9. Let U be a subspace of a vector space V. Then we say that $x, y \in V$ are equivalent if and only if $x - y \in U$. We write

$$(x \sim_U y) \iff (x - y \in U). \tag{3.3}$$

Exercise 3.10. Show that $\sim_U$ is indeed an equivalence relation for every subspace U.

Exercise 3.11. Let U be a subspace of a vector space V over $\mathbb{F}$. Consider equivalence classes in V with respect to the relation $\sim_U$. If a class contains an element x we may denote that class by $x + U$. We define the operations of addition and multiplication by scalars on these equivalence classes by

$$(x + U) + (y + U) := (x + y) + U, \quad \lambda(x + U) := (\lambda x) + U. \tag{3.4}$$

. Show that the operations are well-defined, in other words, the results do not depend on the choices of representatives. Also show that these operations turn the set of equivalence classes into a vector space over $\mathbb{F}$. It is called a quotient space and is denoted by V/U.

Chapters from the textbooks relevant for the lecture:

- Friedberg, Insel, Spence, Sections 1.2, 1.3;
- Roman, pp.17-19, 29-30, 35-38, 41-45, 87-89;
- Axler, ch. 1.