LECTURE 2. BASES, DIMENSION

Throughout this lecture we are going to assume that $\mathbb{F}$ is a field and V is a vector space over $\mathbb{F}$.

1. Linear Span and Linear Independence

Definition 1.1. For a list of vectors $(v_1, \dots, v_n)$ we define its linear span to be the set

$$\boxed{\langle v_1, \dots, v_n \rangle := \{\sum_{j=1}^{n} a_j v_j \mid a_1, \dots, a_n \in \mathbb{F}\}.} \tag{1.1}$$

The expression $\sum_{j=1}^{n} a_j v_j$ is called a linear combination of the vectors $(v_1, \dots, v_n)$. We consider a list of vectors instead of a set to allow a vector appear in that list more than once.

We allow a list to be empty. In that case

$$\langle \emptyset \rangle = \{0\} \tag{1.2}$$

Exercise 1.2. Show that $\langle v_1, \dots, v_n \rangle$ is a subspace of V.

Definition 1.3. A list of vectors $(v_1, \dots, v_n)$ is called linearly dependent if $\exists (a_1, \dots, a_n) \in \mathbb{F}^n$ such that:

(1) $(a_1, \dots, a_n) \neq (0, \dots, 0)$;
(2)

$$\sum_{j=1}^{n} a_j v_j = 0. \tag{1.3}$$

If a list of vectors $(v_1, \dots, v_n)$ is not linearly dependent it is called linearly independent.

A linearly independent list obviously has to be a set.

Exercise 1.4. (Linear Dependence Lemma) Suppose that a list of vectors $\{u_1, \dots, u_m\}$ is linearly dependent, and $u_1 \neq 0$. Show that $\exists 2 \leq j \leq m$ such that the following hold:

(1) $u_j \in \langle u_1, \dots, u_{j-1} \rangle$;
(2) if the j-th term is removed from $(u_1, \dots, u_m)$ then the span of the remaining list equals $\langle u_1, \dots, u_m \rangle$.

Now we can prove an important result that will allow us to introduce the notion of dimension of a vector space later.

Proposition 1.5. ***(Replacement Theorem)*** *Assume that*

$$\langle v_1, \dots, v_n \rangle = V, \tag{1.4}$$

and a subset $\{u_1, \dots, u_m\}$ of V is linearly independent. Then $m \leq n$.

Exercise 1.6. Show this proposition.

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2. A Basis of a Vector Space

Definition 2.1. A set of vectors $\{v_1, \dots, v_n\}$ is called a basis of a vector space V if the following two conditions hold:

(1) A set $\{v_1, \dots, v_n\}$ is linearly independent;
(2) $\langle v_1, \dots, v_n \rangle = V$.

Let us consider other equivalent ways to look at the notion of a basis.

Proposition 2.2. *Let $\{v_1, \dots, v_n\}$ be a set of vectors in V. The following conditions are equivalent:*

(1) *$\{v_1, \dots, v_n\}$ is a basis of V.*

(2) *$\forall x \in V \; \exists! (a_1, \dots, a_n) \in \mathbb{F}^n$ such that*

$$x = \sum_{j=1}^{n} a_j v_j. \tag{2.1}$$

(3) *$\{v_1, \dots, v_n\}$ is a minimal spanning set of V, that is, $\{v_1, \dots, v_n\}$ spans V but any proper subset of $\{v_1, \dots, v_n\}$ does not span V.*

(4) *$\{v_1, \dots, v_n\}$ is a maximal linearly independent set, that is, $\{v_1, \dots, v_n\}$ is linearly independent, but after adding any vector to this set it becomes linearly dependent.*

(5)

$$V = \bigoplus_{j=1}^{n} \langle v_j \rangle. \tag{2.2}$$

Exercise 2.3. Show this proposition.

Let us list corollaries from Proposition 1.5 in the case when there exists a finite basis.

Proposition 2.4. *Let $\{v_1, \dots, v_n\}$ be a basis of V. Then*

(1) *Every list of vectors from V that has the length greater than n is linearly dependent;*

(2) *If $\langle u_1, \dots, u_m \rangle = V$ then $m \geq n$;*

(3) *If $\langle w_1, \dots, w_n \rangle = V$ then the set $\{w_1, \dots, w_n\}$ is a basis of V;*

(4) *If a set $\{x_1, \dots, x_n\}$ is linearly independent then it is a basis of the vector space V;*

(5) *If a set $\{y_1, \dots, y_m\}$ is a basis of V then $m = n$.*

Exercise 2.5. Show this proposition.

3. Dimension of a Vector Space and Its Properties

Definition 3.1. Let $\{v_1, \ldots, v_n\}$ be a basis of V. Then we say that V is a finite-dimensional space, the corresponding notation is $\dim V < \infty$. The number n is called the dimension of the vector space V. We will write $\dim V = n$ in this case.

Note that $\dim\{0\} = 0$.

Exercise 3.2. Let U be a proper subspace of a finite-dimensional vector space V. Show that U is also finite-dimensional and $\dim U < \dim V$.

Next we are going to discuss how we can construct a basis from existing subsets of vectors (Proposition 3.3 and Exercise 3.7).

Proposition 3.3. *Assume that $\dim V = n$ and $\{u_1, \ldots, u_m\}$ is a linearly independent subset of V. Show that there exist vectors $w_1, \ldots, w_{n-m}$ such that $\{u_1, \ldots, u_m, w_1, \ldots, w_{n-m}\}$ is a basis of V.*

Proposition 3.4. *Let U be a subspace of a finite-dimensional vector space V. Then there exists a subspace W such that*

$$V = U \oplus W. \tag{3.1}$$

Exercise 3.5. Prove those two propositions.

Exercise 3.6. Let U be a subspace of a finite-dimensional vector space V. Show that the quotient space V/U is also finite-dimensional and express its dimension using $\dim V$ and $\dim U$.

Exercise 3.7. Assume that $\dim V = n$ and

$$\langle u_1, \ldots, u_m \rangle = V. \tag{3.2}$$

Show that there exists a subset of $\{u_1, \ldots, u_m\}$ that is a basis of V.

Proposition 3.8. *Assume U and W are subspaces of a finite-dimensional vector space V. Then*

$$\boxed{\dim(W + U) = \dim W + \dim U - \dim(W \cap U).} \tag{3.3}$$

Exercise 3.9. Show this proposition.

Exercise 3.10. Assume that $\dim V < \infty$ and

$$V = \bigoplus_{j=1}^{k} U_j. \tag{3.4}$$

Then

$$\dim V = \sum_{j=1}^{k} \dim U_j. \tag{3.5}$$

4. Infinite-dimensional case

Definition 4.1. Assume that S is a subset of V then, by definition, the linear span of S is the following set

$$\text{span}\langle S\rangle := \{\sum_{j=1}^{n} a_j v_j \mid n \in \mathbb{Z}, n \geq 1, v_1, \ldots, v_n \in S, a_1, \ldots, a_n \in \mathbb{F}\} \tag{4.1}$$

Definition 4.2. A subset S of a vector space V is called linearly dependent if $\exists n \in \mathbb{Z}, n \geq 1, \exists v_1, \ldots, v_n \in S, \exists a_1, \ldots, a_n \in \mathbb{F}$ such that:

(1) $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$;
(2)

$$\sum_{j=1}^{n} a_j v_j = 0. \tag{4.2}$$

If a subset S is not linearly dependent then it is called linearly independent.

Definition 4.3. A linearly independent set of vectors S is called a basis of the vector space V if

$$\text{span}\langle S\rangle = V. \tag{4.3}$$

Proposition 4.4. *Suppose that V is a non-zero vector space, a subset M_1 is linearly independent in V and a subset $M_2 \supseteq M_1$ spans V. Then there exists a set S such that $M_1 \subseteq S \subseteq M_2$ and S is a basis of V.*

To show this proposition we may use Zorn's lemma.

Definition 4.5. We say that P is a partially ordered set if for some pairs $(a, b) \in P \times P$ we specify that $a \leq b$ and the following properties hold;

(1) $a \leq a$;
(2) If $a \leq b$ and $b \leq a$ then $a = b$;
(3) If $a \leq b$ and $b \leq c$ then $a \leq c$.

An element d of P is called maximal if there is no element $a \in P, a \neq d$ such that $d \leq a$. A subset $M \subseteq P$ is called totally ordered if for every pair $(a, b) \in P \times P$ either $a \leq b$, or $b \leq a$.

Proposition 4.6 (Zorn's Lemma)**.** *Suppose a partially ordered set P has the property that for every totally ordered subset M of P there exists an element $u \in P$ such that $\forall x \in M$ we have that $x \leq u$. Then the set P contains at least one maximal element.*

One can show that Zorn's Lemma is equivalent to the Zermelo–Fraenkel Axiom of Choice.

Chapters from the textbooks relevant for the lecture:

- Friedberg, Insel, Spence, Sections 1.4 - 1.7;
- Roman, pp.44 - 51;
- Axler, ch. 2.