LECTURE 3. LINEAR MAPS

Throughout this lecture, we will assume that $\mathbb{F}$ is a field, and all vector spaces are defined over that same field. Please consider proof of every proposition in this lecture to be an exercise.

1. Types of Linear Maps

Definition 1.1. Assume that V and U are vector spaces over the same field $\mathbb{F}$. A map $A : V \to U$ is called a linear map (or, a linear transformation, or a homomorphism) if

$$\boxed{A(x+y) = Ax + Ay, \quad A(\lambda x) = \lambda Ax, \quad \forall x, y \in V, \forall \lambda \in \mathbb{F}.} \tag{1.1}$$

The set of all linear maps from V to U is denoted by $\mathrm{Hom}(V, U)$.

Definition 1.2. We may define pointwise addition and multiplication by scalars in $\mathrm{Hom}(V, U)$:

$$(A+B)(x) := Ax + Bx, \quad (\lambda A)(x) := \lambda(Ax), \quad \forall A, B \in \mathrm{Hom}(V,U), \forall x \in V, \forall \lambda \in \mathbb{F}. \tag{1.2}$$

Exercise 1.3. Show that if $A, B \in \mathrm{Hom}(V, U)$ then $A + B, \lambda A \in \mathrm{Hom}(V, U)$.

Exercise 1.4. Show that $\mathrm{Hom}(V, U)$ is a vector space with respect to those operations.

Definition 1.5. We are going to use the following terms and notation:

(1) A linear map from a vector space to itself is called an endomorphism or a linear operator. The set of all linear operators on V is denoted by $\mathrm{End}(V)$. A map $\mathrm{Id} \in \mathrm{End}(V)$ defined by
$$\mathrm{Id}(x) := x, \quad \forall x \in V \tag{1.3}$$
is called the identity map.
(2) A linear map is called a monomorphism or an embedding if it is injective.
(3) A linear map is called an epimorphism if it is surjective.
(4) A linear map is called an isomorphism if it is bijective. The set of all isomorphisms from a vector space V to a vector space U is denoted by $\mathrm{Iso}(V, U)$. If there exists an isomorphism from V to U, we will write $V \cong U$.
(5) A linear operator is called an automorphism if it is bijective. The set of all automorphisms of a vector space V is denoted by $\mathrm{GL}(V)$ (which stands for "General Linear (Group)") or by $\mathrm{Aut}(V)$.

Exercise 1.6. Which of the sets $\mathrm{End}(V)$, $\mathrm{Iso}(V, U)$, $\mathrm{GL}(V)$ are vector spaces with respect to the pointwise addition and the pointwise multiplication by scalars? Is the set of monomorphisms a subspace of $\mathrm{Hom}(V, U$? Is the set of epimorphisms a subspace of $\mathrm{Hom}(V, U)$?

Definition 1.7. Suppose that V, U, W are finite-dimensional spaces, $A \in \mathrm{Hom}(V, U), B \in \mathrm{Hom}(U, W)$. We may also consider an operation of composition of linear maps B and A. We are going to denote this composition of linear maps by BA:

$$BA(x) := B(A(x)), \quad \forall x \in V. \tag{1.4}$$

We use this notation because the operation of the composition of linear operators shares several nice properties with multiplication.

Date: February 23, 2024.

Exercise 1.8. Is $\mathrm{End}(V)$ a group with respect to this operation? Is $\mathrm{GL}(V)$ a group with respect to the operation of composition? Is it always true that

$$A(B+C) = AB + AC, \quad \forall A, B.C \in \mathrm{End}(V)? \tag{1.5}$$

2. Image and Kernel of a Linear Map

Definition 2.1. For an arbitrary $A \in \mathrm{Hom}(V,U)$ we put

$$\ker(A) := \{x \in V \mid Ax = 0\}, \quad \mathrm{im}(A) := \{y \in U \mid \exists x \in V \text{ such that } Ax = y\}. \tag{2.1}$$

The set $\ker(A)$ is called the kernel of A. The set $\mathrm{im}(A)$ is called the image of A.

Proposition 2.2. *For an arbitrary $A \in \mathrm{Hom}(V,U)$ the set $\ker(A)$ is a subspace of the vector space V, $\mathrm{im}(A)$ is a subspace of the vector space U.*

Exercise 2.3. Show that

$$(A \text{ is a monomorpism}) \iff (\ker A = \{0\}). \tag{2.2}$$

Next, we will introduce a version of the fundamental theorem on homomorphisms for vector spaces.

Proposition 2.4. *(The First Isomorphism Theorem) For an arbitrary $A \in \mathrm{Hom}(V,U)$ define the map p from the quotient space $V/\ker(A)$ to the vector space U by*

$$p(x + \ker(A)) := Ax, \quad \forall x \in V. \tag{2.3}$$

Then, p is a well-defined linear map. Moreover, p is an isomorphism between the quotient space $V/\ker(A)$ and $\mathrm{im}(A)$.

Definition 2.5. Assume that V and U are finite-dimensional vector spaces. For a linear map $A \in \mathrm{Hom}(V,U)$ we put

$$\mathrm{rk}(A) := \dim(\mathrm{im}(A)). \tag{2.4}$$

The number $\mathrm{rk}(A)$ is called the rank of the homomorphism A, the number $\dim(\ker(A))$ is called its nullity.

Proposition 2.6. *(The Rank Plus Nullity Theorem) Assume that V and U are finite-dimensional vector spaces. For a linear map $A \in \mathrm{Hom}(V,U)$ we get the identity*

$$\boxed{\mathrm{rk}(A) + \dim(ker(A) = \dim V.} \tag{2.5}$$

3. Isomorphisms

The following two propositions will provide another way to look at isomorphisms.

Proposition 3.1. *A linear map $A \in \mathrm{Hom}(V,U)$ is an isomorphism if and only if $\ker(A) = \{0\}$ and $\mathrm{im}(A) = U$. Then the unverse map A^{-1} is also linear and, therefore, $A^{-1} \in \mathrm{Iso}(U,V)$.*

Proposition 3.2. *Suppose that V and U are vector spaces and $A \in \mathrm{Hom}(V,U)$. Then*

$$(A \in \mathrm{Iso}(V,U)) \iff (\exists B \in \mathrm{Hom}(U,V) \text{ such that } AB = BA = \mathrm{Id}). \tag{3.1}$$

If we can pick a finite basis in a vector space, then we may construct an isomorphism of that space with one of the standard vector spaces $\mathbb{F}^n$.

Proposition 3.3. *Suppose that V is a finite-dimensional space, $\alpha = (v_1, \dots, v_n)$ is a basis of V. We assume that the basis is ordered, i.e., the order of the elements in the set matters. We use the parentheses instead of the curly brackets to stress that we consider an ordered sequence.*

As we know, $\forall x \in V \; \exists! (a_1, \dots, a_n) \in \mathbb{F}^n$ such that

$$x = \sum_{j=1}^{n} a_j v_j. \tag{3.2}$$

The numbers $(a_1, \dots, a_n)$ are called the coordinates of the vector x in the basis α. Let us define a map $\varphi_\alpha : V \to \mathbb{F}^n$ by setting

$$\varphi_\alpha(x) := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}. \tag{3.3}$$

Then the map $\varphi_\alpha \in \mathrm{Iso}(V, \mathbb{F}^n)$.

That claim allows us to provide the following classification of finite-dimensional vector spaces over the same field up to an isomorphism.

Proposition 3.4. *Assume that V and U are finite-dimensional vector spaces. There exists an isomorphism between V and U (or, we may say that V and U are isomorphic) if and only if* $\dim V = \dim U$.

So, to find out whether two finite-dimensional vector spaces are isomorphic, it is sufficient to check their dimensions only. In other words, consider an equivalence relation of being isomorphic on finite-dimensional vector spaces over a fixed vector space $\mathbb{F}$. Then, every equivalence class consists of all finite-dimensional vector spaces of the same dimension, including $\mathbb{F}^n$ for some n.

Note that to figure out whether a linear map between two vector spaces of the same finite dimension is an isomorphism it is sufficient to check either injectivity or surjectivity.

Proposition 3.5. *Assume that V and U are finite-dimensional vector spaces and* $\dim V = \dim U$.

a) If $A \in Hom(V, U)$ is a monomorphism then A is an isomorphism.

b) If $A \in Hom(V, U)$ is an epimorphism then A is an isomorphism.

4. The Matrix of a Linear Map

Proposition 4.1. *Assume that $(v_1, \dots, v_n)$ is a basis of a vector space V.*

a) Then for every list of vectors $y_1, \dots, y_n \in U$ there exists a unique map $B \in \mathrm{Hom}(V, U)$ such that $Bv_j = y_j$, $\forall 1 \le j \le n$.

b) Every map $A \in \mathrm{Hom}(V, U)$ is completely determined by the ordered set of images $Av_1, \dots, Av_n$.

The next definition establishes how we can represent linear maps via matrices.

Definition 4.2. Assume that $\alpha = (v_1, \dots, v_n)$ is an ordered basis of a vector space V, $\beta = (u_1, \dots, u_m)$ is an ordered basis of a vector space U, $A \in \mathrm{Hom}(V, U)$. Then there exists a list of numbers (a_i^j) such that

$$\boxed{Av_i = \sum_{j=1}^{m} a_i^j u_j, \quad \forall 1 \le i \le n.} \tag{4.1}$$

The matrix filled out with (a_i^j)

$$\begin{pmatrix} a_1^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \dots & a_n^m \end{pmatrix} \tag{4.2}$$

is called the matrix of the map A (in the bases α and β) and is denoted by the same letter A or by $[A]^{\beta}_{\alpha}$. The set of $m \times n$ matrices with the entries from the field $\mathbb{F}$ is denoted by $\mathrm{Mat}(m \times n, \mathbb{F})$.

Exercise 4.3. Assume that V and U are finite-dimensional vector spaces with a fixed ordered basis in each of them, $A, B \in \mathrm{Hom}(V, U)$. Express the entries of the matrices of the linear maps $A + B$ and λA using the entries of A and B.

Exercise 4.4. It is obvious that the set $\mathrm{Mat}(m \times n, \mathbb{F})$ is a vector space with respect to entrywise addition and entrywise multiplication by scalars. Suppose that V and U are finite-dimensional vector spaces. Show that the vector spaces $Hom(V, U)$ and $Mat(\mathrm{dim}U \times \mathrm{dim}V, \mathbb{F})$ are isomorphic.

Exercise 4.5. Assume that V and U are finite-dimensional vector spaces. Show that

$$\boxed{\dim(\mathrm{Hom}(V, U) = \dim V \cdot \dim U.} \tag{4.3}$$

Now, we will establish the rule of multiplication of matrices that corresponds to the composition of linear maps.

Definition 4.6. (The Row-Column rule) Suppose that V, U, W are finite-dimensional spaces with bases α, β, γ respectively, $A \in \mathrm{Hom}(V, U), B \in \mathrm{Hom}(U, W)$. We define the product of matrices via the formula

$$\boxed{[B]^{\gamma}_{\beta} \cdot [A]^{\beta}_{\alpha} := [BA]^{\gamma}_{\alpha}.} \tag{4.4}$$

Exercise 4.7. Assume that X is an arbitrary $m \times n$ matrix, Y is an arbitrary $p \times m$ matrix. Show that Definition 4.6 defines a unique product $Y \cdot X$, which is an $p \times n$ matrix.

Using the Row-Column rule notation, we may rewrite the Definition 4.2 as symbols satisfying

$$(Av_1, \ldots, Av_n) = (u_1, \ldots, u_m) \begin{pmatrix} a_1^1 & \ldots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \ldots & a_n^m \end{pmatrix}. \tag{4.5}$$

Chapters from the textbooks relevant to the lecture:

- Friedberg, Insel, Spence, Sections 2.1 - 2.5;
- Roman, pp.59 - 79;
- Axler, ch. 3.