# LECTURE 5. TENSOR PRODUCTS OF VECTOR SPACES. DETERMINANT.

Throughout this lecture, we will assume that $\mathbb{F}$ is a field, and vector spaces are defined over that same field. Please consider proof of every proposition in this lecture to be an exercise.

## 1. Tensor Products of Vector Spaces

**Definition 1.1.** Assume that $V$ and $U$ are vector spaces over a field $\mathbb{F}$. Denote by $Free(V,U)$ the set of all formal finite linear combinations elements of $V \times U$:

$$Free(V,U) := \left\{ \sum_{i=1}^{n} a_i(x_i, y_i) \mid n \in \mathbb{Z}_+,\ x_i \in V, y_i \in U, a_i \in \mathbb{F}, \forall 1 \le i \le n \right\}. \tag{1.1}$$

(Note that the pairs $(x_i, y_i)$ are simply symbols here. Probably it would be less confusing if we wrote $Free(V,U)$ as

$$\left\{ \sum_{i=1}^{n} a_i e_{(x_i,y_i)} \mid n \in \mathbb{Z}_+,\ x_i \in V, y_i \in U, a_i \in \mathbb{F}, \forall 1 \le i \le n \right\} \tag{1.2}$$

but then we would get more cumbersome expressions further in this definition.)

The set $Free(V,U)$ has the natural structure of a vector space. Now we may define a subspace $Ident(V,U) \subset Free(V,U)$ as

$$\begin{aligned} Ident(V,U) = \operatorname{span}\langle \{(ax+bz,y) - a(x,y) - b(z,y) \,|\, x,z \in V, y \in U, a,b \in \mathbb{F}\} \cup \\ \{(x, ay+bt) - a(x,y) - b(x,t) \,|\, x \in V, y,t \in U, a,b \in \mathbb{F}\} \rangle. \end{aligned} \tag{1.3}$$

The tensor product of vector spaces $V$ and $U$ is, by definition, the quotient space $Free(V,U)$ by $Ident(V,U)$. We denote the tensor product of vector spaces $V$ and $U$ by $V \otimes U$. Also we denote by

$$\sum_{i=1}^{n} a_i x_i \otimes y_i \tag{1.4}$$

the equivalence class that contains the element $\sum_{i=1}^{n} a_i(x_i, y_i)$. A map $\operatorname{emb} : V \times U \to V \otimes U$ is defined by $\operatorname{emb}((v,u)) := v \otimes u$. The elements of $V \otimes U$ are often called tensors.

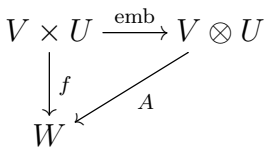
For example, the element $(a \cdot x + 0 \cdot z, y) - a(x,y) - 0 \cdot (z,y) \in Ident(V,U)$. Therefore, $(ax) \otimes y = a(x \otimes y)$, $\forall a \in \mathbb{F}$. In particular, $0 \otimes y = 0$.

The following statement may be considered as a property characterizing the tensor product of two vector spaces.

**Proposition 1.2.** *Suppose that $V, U$ and $W$ are vector spaces over a field $\mathbb{F}$, and a map $f : V \times U \to W$ satisfies the following conditions:*

$$\begin{aligned} f(ax+bz,y) &= af(x,y) + bf(z,y), \\ f(x,ay+bt) &= af(x,y) + bf(x,t), \quad \forall x,z \in V, \forall y,t \in U, \forall a,b \in \mathbb{F}. \end{aligned} \tag{1.5}$$

*Then there exists a unique map $A \in \operatorname{Hom}(V \otimes U, W)$ such that $A \circ \operatorname{emb} = f$.*

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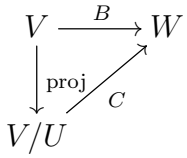
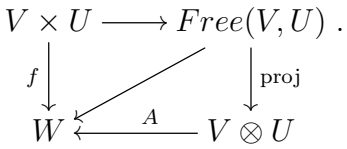
To prove Proposition 1.2, we will establish the following lemma first.

**Lemma 1.3.** *Suppose that $V, W$ are vector spaces, $U$ is a subspace of $V$, by* $\mathrm{proj} : V \to V/U$ *we denote the canonical projection, and $B \in \mathrm{Hom}(V, W)$. Then, the existence of a map $C \in \mathrm{Hom}(V/U, W)$ such that $C \circ \mathrm{proj} = B$ is equivalent to the condition $U \subseteq \ker(B)$.*




Returning to the proof of Proposition 1.2, we note that the diagram defines the map $g : Free(V, U) \to W$. Then, we may apply Lemma 1.3 to the lower left triangle of the diagram




**Proposition 1.4.** *Assume that $V$ and $U$ are finite-dimensional vector spaces over a field $\mathbb{F}$, $\{v_1, \dots, v_n\}$ is a basis of $V$, $\{u_1, \dots, u_m\}$ is a basis of $U$. Then*

$$\{v_i \otimes u_j \,|\, 1 \le i \le n, 1 \le j \le m\} \tag{1.6}$$

*is a basis of $V \otimes U$.*

One may use the result of Proposition 1.2 in the proof of linear independence of the set in (1.6).

**Proposition 1.5.** *Assume that $V$ and $U$ are finite-dimensional vector spaces. Define $\varphi : V^* \otimes U \to \mathrm{Hom}(V, U)$ by*

$$\varphi(f \otimes u)(v) := f(v)u, \quad \forall v \in V, u \in U, f \in V^*. \tag{1.7}$$

*The map $\varphi$ is well-defined and $\varphi \in Iso(V^* \otimes U, \mathrm{Hom}(V, U))$.*

Note that in the Proposition 1.5, we have defined the map only on the tensors of the form $v \otimes u$. That is not uncommon. One may say that we follow here Proposition 1.2.

As a special case of Proposition 1.5, we may obtain an important corollary.

**Proposition 1.6.** *Assume that $V$ is a finite-dimensional vector space. Then the vector spaces $V \otimes V^*$ and $\mathrm{End}(V)$ are canonically isomorphic.*

**Definition 1.7.** Assume that $V, U$ are finite-dimensional vector spaces over a field $\mathbb{F}$, $A \in \mathrm{End}(V)$, $B \in \mathrm{End}(U)$. Define the tensor product of linear operators $A \otimes B \in \mathrm{End}(V \otimes U)$ by the formula

$$(A \otimes B)(v \otimes u) := (Av) \otimes (Bu) \tag{1.8}$$

on the tensors of the form $v \otimes u$ and then extend by linearity. By definition, we put

$$A^{\otimes k} := \underbrace{A \otimes A \otimes \cdots \otimes A}_{k \text{ times}} \tag{1.9}$$

**Exercise 1.8.** Show that $A \otimes B$ is well-defined.

**Exercise 1.9.** Suppose that $A, B \in \mathrm{End}(V)$. Is it always true that

$$(AB)^{\otimes k} = ((A)^{\otimes k})((B)^{\otimes k}) \tag{1.10}$$

**Exercise 1.10.** Assume that $V, U$ are finite-dimensional vector spaces over a field $\mathbb{F}$, $\{v_1, \dots, v_n\}$ is a basis of $V$, $\{u_1, \dots, u_m\}$ is a basis of $U$, $A \in \mathrm{End}(V)$, its matrix in the basis $\{v_1, \dots, v_n\}$ is

$$(a_i^j)_{1 \le i \le n, 1 \le j \le n}, \tag{1.11}$$

$B \in \mathrm{End}(U)$, its matrix in the basis and $\{u_1, \dots, u_m\}$ is

$$(b_i^j)_{1 \le i \le m, 1 \le j \le m}. \tag{1.12}$$

Find the matrix of $A \otimes B$ in the basis

$$\{v_1 \otimes u_1, v_1 \otimes u_2, \dots, v_1 \otimes u_m, v_2 \otimes u_1, \dots, v_n \otimes u_m\}. \tag{1.13}$$

## 2. Symmetric Group

**Definition 2.1.** A map $\rho : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is called a permutation (of the set $\{1, 2, \dots, n\}$) if $\rho$ is a bijection. The set of all permutations of the set $\{1, 2, \dots, n\}$ is called a symmetric group and is denoted by $S_n$. The composition $\rho \circ \sigma$ of two permutations $\rho, \sigma \in S_n$ is denoted by $\rho\sigma$. For an element $\rho \in S_n$ we may use a two-row notation:

$$\rho = \begin{pmatrix} 1 & 2 & \dots & n \\ \rho(1) & \rho(2) & \dots & \rho(n). \end{pmatrix} \tag{2.1}$$

A more common notation is a cycle notation. It is defined in the following way. Let us fix an arbitrary permutation $\rho \in S_n$. We introduce an equivalence relation on the set $\{1, 2, \dots, n\}$ by the following rule:

$$(i \sim j) \quad \Longleftrightarrow \quad (\exists k \in \mathbb{Z}_+ \text{ such that } \rho^k(i) = j), \tag{2.2}$$

where

$$\rho^k := \underbrace{\rho \circ \rho \circ \dots \circ \rho}_{k \text{ times}}. \tag{2.3}$$

One may check that it is indeed an equivalence relation. For each equivalence class that consists of, say, $m$ elements we pick a representative $j$ and write a cycle $(j, \rho(j), \rho^2(j), \dots, \rho^{m-1}(j))$ in the cycle notation of $\rho$. If $m = 1$, we usually omit the corresponding one-element cycle. Here is an example of cycle notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 1 & 4 & 3 \end{pmatrix} = (1, 5, 4)(3, 6). \tag{2.4}$$

One may show that cycle notation is well-defined. A cycle of length 2 is called a transposition.

Here is an example of the product of two permutations written in cycle notation:

$$((1, 2)(3, 4, 5))((2, 3)(1, 5)) = (1, 3)(2, 4, 5). \tag{2.5}$$

**Exercise 2.2.** Show that $S_n$ is a group with respect to composition and find its order.

One may compare the following statement with the change of basis for linear operators.

**Exercise 2.3.** Show that for every permutation $\rho \in S_n$ and for every subset $\{i_1, i_2, \dots, i_k\}$ of the set $\{1, 2, \dots, n\}$ the following equality holds

$$\boxed{\rho \circ (i_1, i_2, \dots, i_k) \circ \rho^{-1} = (\rho(i_1), \rho(i_2), \dots, \rho(i_k)),} \tag{2.6}$$

, where the LHS is the composition of three permutations such that the second permutation is a cycle of length $k$.

**Exercise 2.4.** Suppose that $\rho \in S_n$. Consider a map $\psi_\rho : S_n \to S_n$ defined by

$$\psi_\rho(\sigma) := \rho\sigma\rho^{-1}. \tag{2.7}$$

Show that $\psi_\rho$ is an automorphism of the group $S_n$.

**Exercise 2.5.** Show that every permutation $\rho \in S_n$ can be represented as a product of some transpositions.

**Proposition 2.6.** *For every positive integer $n$ there exists a unique map* $\mathrm{sgn} : \{1, 2, \dots, n\} \to \{1, -1\}$ *such that*

(1) $\mathrm{sgn}((i, j)) = -1, \quad \forall 1 \le i < j \le n;$
(2) $\mathrm{sgn}(\rho\sigma) = \mathrm{sgn}(\rho)\mathrm{sgn}(\sigma).$

*The value* $\mathrm{sgn}(\rho)$ *is called the sign of the permutation* $\rho$.

To prove this proposition, one may consider the ratio

$$\frac{\prod_{1\le i<j\le n}(x_{\rho(i)} - x_{\rho(j)})}{\prod_{1\le i<j\le n}(x_i - x_j)}. \tag{2.8}$$

## 3. Symmetric and Exterior Powers

**Definition 3.1.** Let $V$ be a vector space over $\mathbb{F}$. Define the subspaces of $k$-th symmetric and exterior powers of $V$, respectively, $S^k(V)$ and $\Lambda^k(V)$ of

$$V^{\otimes k} := \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ times}} \tag{3.1}$$

as

$$S^k(V) := \mathrm{span}\langle \sum_{\rho\in S_k} x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)} \,|\, x_1, x_2, \dots, x_k \in V\rangle, \tag{3.2}$$

$$\Lambda^k(V) := \mathrm{span}\langle \sum_{\rho\in S_k} \mathrm{sgn}(\rho) x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)} \,|\, x_1, x_2, \dots, x_k \in V\rangle. \tag{3.3}$$

We use the notation

$$x_1 \wedge x_2 \wedge \cdots \wedge x_k := \frac{1}{k!} \sum_{\rho\in S_k} \mathrm{sgn}(\rho) x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)}. \tag{3.4}$$

**Exercise 3.2.** Suppose that $\dim V = n$. Find $\dim S^k(V)$ and $\dim \Lambda^k(V)$.

**Definition 3.3.** Suppose that $A \in \mathrm{End}(V)$. One may show that

$$\forall z \in S^k(V) \text{ we get that } A^{\otimes k}(z) \in S^k(V), \tag{3.5}$$

$$\forall z \in \Lambda^k(V) \text{ we get that } A^{\otimes k}(z) \in \Lambda^k(V). \tag{3.6}$$

We are going to denote by $S^k(A) \in \mathrm{End}(S^k(V))$ and $\Lambda^k(A) \in \mathrm{End}(\Lambda^k(V))$ the restrictions of $A^{\otimes k}$ to the subspaces $S^k(V)$ and $\Lambda^k(V)$, respectively.

## 4. Determinant

**Definition 4.1.** Assume that $V$ is a finite-dimensional vector space, $A \in \mathrm{End}(V)$. Note that $\dim(\Lambda^{\dim V}(V)) = 1$. Therefore, $\Lambda^{\dim V}(A)$ is a multiple of the identity map. We will define the determinant $\det(A)$ of the linear operator $A$ by the following equality

$$\boxed{\Lambda^{\dim V}(A) = \det(A) \cdot \mathrm{Id}.} \tag{4.1}$$

**Proposition 4.2.** *Suppose that* $\dim V < \infty$ *and* $A, B \in \mathrm{End}(V)$. *Then*

$$\boxed{\det(AB) = \det(A) \cdot \det(B).} \tag{4.2}$$

**Proposition 4.3.** *Suppose that* $\dim V = n < \infty$ *and* $A \in \mathrm{End}(V)$ *has a matrix* $(a_i^j)_{1 \le i,j \le n}$ *in some basis. Then*

$$\boxed{\det(A) = \sum_{\rho \in S_n} \mathrm{sgn}(\rho) \prod_{j=1}^{n} a^j_{\rho(j)}.} \tag{4.3}$$

Chapters from the textbooks relevant to the lecture:

- Roman, pp. 363-378, 393-395.