LECTURE 6. TENSOR PRODUCTS OF VECTOR SPACES. DETERMINANT.

Throughout this lecture, we will assume that $\mathbb{F}$ is a field, and vector spaces are defined over that same field. Please consider proof of every proposition in this lecture to be an exercise.

1. Symmetric Group

Definition 1.1. A map $\rho : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ is called a permutation (of the set $\{1, 2, \dots, n\}$) if ρ is a bijection. The set of all permutations of the set $\{1, 2, \dots, n\}$ is called a symmetric group and is denoted by S_n. The composition $\rho \circ \sigma$ of two permutations $\rho, \sigma \in S_n$ is denoted by $\rho\sigma$. For an element $\rho \in S_n$ we may use a two-row notation:

$$\rho = \begin{pmatrix} 1 & 2 & \dots & n \\ \rho(1) & \rho(2) & \dots & \rho(n). \end{pmatrix} \tag{1.1}$$

A more common notation is a cycle notation. It is defined in the following way. Let us fix an arbitrary permutation $\rho \in S_n$. We introduce an equivalence relation on the set $\{1, 2, \dots, n\}$ by the following rule:

$$(i \sim j) \quad \Longleftrightarrow \quad (\exists k \in \mathbb{Z}_+ \text{ such that } \rho^k(i) = j), \tag{1.2}$$

where

$$\rho^k := \underbrace{\rho \circ \rho \circ \dots \circ \rho}_{k \text{ times}}. \tag{1.3}$$

One may check that it is indeed an equivalence relation. For each equivalence class that consists of, say, m elements we pick a representative j and write a cycle $(j, \rho(j), \rho^2(j), \dots, \rho^{m-1}(j))$ in the cycle notation of ρ. If $m = 1$, we usually omit the corresponding one-element cycle. Here is an example of cycle notation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 2 & 6 & 1 & 4 & 3 \end{pmatrix} = (1, 5, 4)(3, 6). \tag{1.4}$$

One may show that cycle notation is well-defined. A cycle of length 2 is called a transposition.

Here is an example of the product of two permutations written in cycle notation:

$$((1, 2)(3, 4, 5))((2, 3)(1, 5)) = (1, 3)(2, 4, 5). \tag{1.5}$$

Exercise 1.2. Show that S_n is a group with respect to composition and find its order.

One may compare the following statement with the change of basis for linear operators.

Exercise 1.3. Show that for every permutation $\rho \in S_n$ and for every subset $\{i_1, i_2, \dots, i_k\}$ of the set $\{1, 2, \dots, n\}$ the following equality holds

$$\boxed{\rho \circ (i_1, i_2, \dots, i_k) \circ \rho^{-1} = (\rho(i_1), \rho(i_2), \dots, \rho(i_k)),} \tag{1.6}$$

, where the LHS is the composition of three permutations such that the second permutation is a cycle of length k.

Exercise 1.4. Suppose that $\rho \in S_n$. Consider a map $\psi_\rho : S_n \to S_n$ defined by

$$\psi_\rho(\sigma) := \rho\sigma\rho^{-1}. \tag{1.7}$$

Show that ψ_ρ is an automorphism of the group S_n.

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Exercise 1.5. Show that every permutation $\rho \in S_n$ can be represented as a product of some transpositions.

Proposition 1.6. *For every positive integer n there exists a unique map* $\mathrm{sgn} : S_n \to \{1, -1\}$ *such that*

(1) $\mathrm{sgn}((i,j)) = -1, \quad \forall 1 \le i < j \le n;$
(2) $\mathrm{sgn}(\rho\sigma) = \mathrm{sgn}(\rho)\mathrm{sgn}(\sigma).$

The value $\mathrm{sgn}(\rho)$ *is called the sign of the permutation* ρ.

To prove this proposition, one may consider the ratio

$$\frac{\prod_{1\le i<j\le n}(x_{\rho(i)} - x_{\rho(j)})}{\prod_{1\le i<j\le n}(x_i - x_j)}. \tag{1.8}$$

2. Symmetric and Exterior Powers

Definition 2.1. Let V be a vector space over $\mathbb{F}$. Define the subspaces of k-th symmetric and exterior powers of V, respectively, $S^k(V)$ and $\Lambda^k(V)$ of

$$V^{\otimes k} := \underbrace{V \otimes V \otimes \cdots \otimes V}_{k \text{ times}} \tag{2.1}$$

as

$$S^k(V) := \mathrm{span}\langle \sum_{\rho\in S_k} x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)} \,|\, x_1, x_2, \ldots, x_k \in V\rangle, \tag{2.2}$$

$$\Lambda^k(V) := \mathrm{span}\langle \sum_{\rho\in S_k} \mathrm{sgn}(\rho) x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)} \,|\, x_1, x_2, \ldots, x_k \in V\rangle. \tag{2.3}$$

We use the notation

$$x_1 \wedge x_2 \wedge \cdots \wedge x_k := \frac{1}{k!} \sum_{\rho\in S_k} \mathrm{sgn}(\rho) x_{\rho(1)} \otimes x_{\rho(2)} \otimes \cdots \otimes x_{\rho(k)}. \tag{2.4}$$

Exercise 2.2. Suppose that $\dim V = n$. Find $\dim S^k(V)$ and $\dim \Lambda^k(V)$.

Definition 2.3. Suppose that $A \in \mathrm{End}(V)$. One may show that

$$\forall z \in S^k(V) \text{ we get that } A^{\otimes k}(z) \in S^k(V), \tag{2.5}$$

$$\forall z \in \Lambda^k(V) \text{ we get that } A^{\otimes k}(z) \in \Lambda^k(V). \tag{2.6}$$

We are going to denote by $S^k(A) \in \mathrm{End}(S^k(V))$ and $\Lambda^k(A) \in \mathrm{End}(\Lambda^k(V))$ the restrictions of $A^{\otimes k}$ to the subspaces $S^k(V)$ and $\Lambda^k(V)$, respectively.

3. Determinant

Definition 3.1. Assume that V is a finite-dimensional vector space, $A \in \mathrm{End}(V)$. Note that $\dim(\Lambda^{\dim V}(V)) = 1$. Therefore, $\Lambda^{\dim V}(A)$ is a multiple of the identity map. We will define the determinant $\det(A)$ of the linear operator A by the following equality

$$\boxed{\Lambda^{\dim V}(A) = \det(A) \cdot \mathrm{Id}.} \tag{3.1}$$

Proposition 3.2. *Suppose that* $\dim V < \infty$ *and* $A, B \in \mathrm{End}(V)$. *Then*

$$\boxed{\det(AB) = \det(A) \cdot \det(B).} \tag{3.2}$$

Proposition 3.3. *Suppose that* $\dim V = n < \infty$ *and* $A \in \mathrm{End}(V)$ *has a matrix* $(a_i^j)_{1 \le i,j \le n}$ *in some basis. Then*

$$\boxed{\det(A) = \sum_{\rho \in S_n} \mathrm{sgn}(\rho) \prod_{j=1}^{n} a^j_{\rho(j)}.} \tag{3.3}$$

4. Decomposition of polynomials

Definition 4.1. A polynomial in a variable x is a formal sum of the form

$$p = \sum_{j=0}^{k} a_j x^j, \quad k \ge 0,\ a_j \in \mathbb{F}. \tag{4.1}$$

We assume that two sums of that form are equal if they differ only by some monomials with zero coefficients. If $a_k \neq 0$ we say that the degree of the polynomial p $\deg(p) := k$. In that case, we are going to call a_k the leading coefficient of the polynomial p. If all coefficients of the polynomial equal zero then we put $\deg(0) := -\infty$. The set of polynomials is a vector space with respect to standard addition and scalar multiplication. We denote that vector space by $\mathbb{F}[x]$. Also, we put

$$\mathbb{F}_n[x] := \{p \in \mathbb{F}[x] \mid \deg(p) \le n\}. \tag{4.2}$$

With every polynomial $p \in \mathbb{F}[x]$ we associate a function $p : \mathbb{F} \to \mathbb{F}$.

Note that $\forall a \in \mathbb{F}, a \neq 0$ we have that $\deg(a) = 0$. Next we may define Euclidean division for polynomials.

Proposition 4.2. *For every two polynomials* $f, g \in \mathbb{F}[x], g \neq 0$ *there exist unique polynomials* $q, r \in \mathbb{F}[x]$ *such that the following conditions hold:*

(1) $f = q \cdot g + r$;
(2) $\deg(r) < \deg g$.

Definition 4.3. We will say that a polynomial $g \in \mathbb{F}[x], g \neq 0$ divides a polynomial $f \in \mathbb{F}[x]$ if there exists a polynomial $q \in \mathbb{F}[x]$ such that $f = g \cdot q$. We are going to write $g \mid f$. Note that $\deg(f) \ge \deg(g)$ in that case.

For arbitrary polynomials $f_1, \ldots, f_m \in \mathbb{F}[x]$ we are going to denote their greatest common divisor by $GCD(f_1, \ldots, f_m)$. The GCD is determined up to multiplication by a non-zero scalar, so we will assume that its leading coefficient equals one to ensure uniqueness.

Example 4.4. Suppose that

$$p_1(x) = x^3 - x, \quad p_2(x) = x^3 - x^2 + x - 1, \quad p_3(x) = x^2 - 1. \tag{4.3}$$

Then

$$p_3 \mid p_1, \quad p_3 \nmid p_2, \quad GCD(p_1, p_2, p_3) = x - 1. \tag{4.4}$$

Exercise 4.5. Show that

$$((x - a) \mid f) \quad \Longleftrightarrow \quad (f(a) = 0). \tag{4.5}$$

Now we are going to state an important property that has its analogs for integers and, more generally, for principal ideals domains.

Proposition 4.6 (Bézout's identity for polynomials)**.** *Suppose that $f_1, \dots, f_m \in \mathbb{F}[x]$ and $g = GCD(f_1, \dots, f_m)$. Then there exist $p_1, \dots, p_m \in \mathbb{F}[x]$ such that*

$$g = \sum_{j=1}^{m} p_j \cdot f_j. \tag{4.6}$$

To prove that result, we may consider the set

$$I := \{\sum_{j=1}^{m} p_j \cdot f_j \mid p_j \in \mathbb{F}[x]\} \tag{4.7}$$

and put g to be a non-zero element of I of minimal degree. Again it is defined up to multiplication by a non-zero scalar.

Proposition 4.6 allows us to prove the following lemma.

Lemma 4.7. *Suppose that $g, p, q \in \mathbb{F}[x]$, $g \mid pq$ and $GCD(g, p) = 1$. Then $g \mid q$.*

Definition 4.8. A polynomial f is called irreducible if

(1) $\deg(f) \geq 1$;
(2) for every $g, q \in \mathbb{F}[x]$ such that $f = g \cdot q$ we have that either $\deg(g) = 0$ or $\deg(q) = 0$.

From Lemma 4.7 we may deduce the unique factorization property for polynomials in one variable.

Proposition 4.9. *Every polynomial $f \in \mathbb{F}[x]$ may be factored into a product of a non-zero constant and a finite number of irreducible polynomials:*

$$f = a \prod_{j=1}^{m} q_j, \quad a \in \mathbb{F}, q_j \in \mathbb{F}[x]. \tag{4.8}$$

This decomposition is unique up to the permutation of the factors and the multiplication of the factors by non-zero constants whose product is 1. Again we may assume that the leading coefficient of every q_j eguals one to ensure uniqueness.

Example 4.10. We have the following decompositions

$$2x^3 + x \in \mathbb{R}[x], \quad 2x^3 + x = 2x(x^2 + 1); \tag{4.9}$$

$$2x^3 + x \in \mathbb{C}[x], \quad 2x^3 + x = 2x(x + i)(x - i). \tag{4.10}$$

We are going to take for granted the next claim.

Proposition 4.11 (Fundamental Theorem of Algebra)**.** *Every irreducible complex polynomial equals* $(x - z)$ *for some* $z \in \mathbb{C}$*.*

From the complex case we may deduce the analogous result for the field $\mathbb{R}$.

Proposition 4.12. *Every irreducible real polynomial equals either* $(x - a)$ *for some* $a \in \mathbb{R}$*, or* $x^2 + bx + c$*, where* $b^2 - 4c < 0$*.*

It is useful to consider the conjugation in the field $\mathbb{C}$ to prove this proposition.

Chapters from the textbooks relevant to the lecture:

- Roman, pp. 363-378, 393-395.