LECTURE 8. EIGENVALUES AND EIGENVECTORS. PRIMARY DECOMPOSITION THEOREM.

Please consider proof of every proposition in this lecture to be an exercise. The notation $\mathbb{F}$ stands for a field.

1. Eigenvalues and Eigenvectors.

In this section, we will consider the case when an invariant subspace is one-dimensional.

Definition 1.1. Suppose that A is a linear operator in a vector space V. An element $v \in V$ is called an eigenvector of the endomorphism A if:

(1) $v \neq 0$;
(2) $\exists \lambda \in \mathbb{F}$ such that $Av = \lambda v$.

The number λ is called an eigenvalue of the linear operator A. The set of all eigenvalues of A is called the spectrum of A and is denoted by $\mathrm{Spec}(A)$.

Exercise 1.2. Find $\mathrm{Spec}(\frac{d}{dx})$ in the vector space $\mathbb{R}_n[x]$.

Now, we will formulate a possible algorithm explaining how to find all eigenvalues of a given linear operator.

Proposition 1.3. *Suppose that $A \in \mathrm{End}(V)$. Then*

$$(\lambda \text{ is an eigenvalue of } A) \iff (\lambda \text{ is a root of the characteristic polynomial } \chi_A) \tag{1.1}$$

Definition 1.4. Suppose that A is a linear operator in a vector space V and λ is an eigenvalue of A. Put

$$V_\lambda := \{x \in V \mid Ax = \lambda x\}. \tag{1.2}$$

V_λ is called an eigenspace of A. We see that V_λ is a subspace of the vector space V. Its dimension $\dim V_\lambda$ is called the geometric multiplicity of λ.

Note that V_λ consists of zero vector and all eigenvectors with the eigenvalue λ.

Assume that we know all roots of the characteristic polynomial of A, then we may find all eigenvectors of the linear operator A by solving equations $(A - \lambda \mathrm{Id})(x) = 0$ for every root λ. If $\dim V < \infty$ and we fix a basis of V, then for every fixed λ, the equation $(A - \lambda \mathrm{Id})(x) = 0$ is a homogeneous system of linear equations. And, therefore,

$$\dim V_\lambda = \dim V - \mathrm{rk}(A - \lambda \mathrm{Id}). \tag{1.3}$$

Question 1.5. How can we describe all linear operators $A \in \mathrm{End}(V)$ in a finite-dimensional vector space for which the geometric multiplicity of some eigenvalue equals $\dim V$?

Definition 1.6. Suppose that A is a linear operator in a finite-dimensional vector space V. Consider the factorization of the characteristic polynomial χ_A into irreducible polynomials. The multiplicity of the factor $(x - \lambda)$ in this product is called the algebraic multiplicity of λ.

Proposition 1.7. *Suppose that A is a linear operator in a finite-dimensional vector space V, λ is an eigenvalue of A. We claim that the geometric multiplicity of λ is not greater than its algebraic multiplicity.*

Question 1.8. Can the geometric multiplicity of some eigenvalue be equal to its algebraic multiplicity?

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Exercise 1.9. Suggest an example of a linear operator such that the geometric multiplicity of some eigenvalue is strictly less than its algebraic multiplicity.

2. Diagonalization.

Proposition 2.1. *Assume that $v_1, \dots, v_k$ are eigenvalues of a linear operator A with distinct eigenvalues, then the set $\{v_1, \dots, v_k\}$ is linearly independent.*

From Proposition 2.1, we may deduce a useful result about the corresponding eigenspaces.

Proposition 2.2. *Suppose that A is a linear operator in a finite-dimensional vector space V, $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of A, then the sum of the corresponding eigenspaces*

$$V_{\lambda_1} + \dots + V_{\lambda_k} \tag{2.1}$$

is a direct sum.

Definition 2.3. A square matrix

$$B = (b_i^j)_{1 \le i,j \le n} \in \mathrm{Mat}(n \times n, \mathbb{F}) \tag{2.2}$$

is called diagonal if $b_i^j = 0$ for all $i \neq j$:

$$\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}. \tag{2.3}$$

A linear operator A in a finite-dimensional vector space V is called diagonalizable if and only if there exists a basis of V in which the matrix of A is diagonal.

Exercise 2.4. Show that a linear operator defined by the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is not diagonalizable.

Definition 2.5. We say that a polynomial $p \in \mathbb{F}[x]$ splits if it can be represented as

$$p(x) = c \cdot \prod_{j=1}^{k} (x - a_j)^{m_j} \tag{2.4}$$

for some $a_1, \dots, a_k, c \in \mathbb{F}$, $m_1, \dots, m_k \in \mathbb{Z}_+$.

Now we will state an obvious property of diagonalizable linear operators.

Proposition 2.6. *If a linear operator A in a finite-dimensional vector space V is diagonalizable then its characteristic polynomial splits.*

Exercise 2.7. Suppose that the characteristic polynomial of an operator A in a finite-dimensional vector space V splits. Does A have to be diagonalizable?

Using Proposition 2.1, we can obtain a sufficient condition for diagonalizability.

Proposition 2.8. *If the characteristic polynomial of a linear operator A in a finite-dimensional vector space V has $\dim V$ distinct roots then A is diagonalizable.*

Also, we may easily formulate a statement about triangulation.

Definition 2.9. A square matrix

$$B = (b_i^j)_{1 \le i,j \le n} \in \mathrm{Mat}(n \times n, \mathbb{F}) \tag{2.5}$$

is called upper triangular if $b_i^j = 0$ for all $i < j$.

Proposition 2.10 (Schur's Theorem)**.** *Suppose that A is a linear operator in a finite-dimensional vector space V and its characteristic polynomial splits, then there exists a basis of V in which the matrix of A is upper triangular.*

In the proof of Schur's Theorem, we may recall how we connected the characteristic polynomials of a linear operator and its restriction to an invariant subspace in the previous lecture.

3. Primary Decomposition Theorem

Definition 3.1. Suppose that A is a linear operator in a vector space V, λ is an eigenvalue of A. Put

$$V^{\lambda} := \{x \in V \mid \exists k \in \mathbb{Z}_+ \text{ such that } (A - \lambda \cdot \mathrm{Id})^k(x) = 0\}. \tag{3.1}$$

V^{λ} is called a generalized eigenspace of A. We immediately get that V^{λ} is a subspace of the vector space V.

Question 3.2. Can V^{λ} coincide with V_{λ}?

Proposition 3.3. *Suppose that A is a linear operator in a vector space V, λ is an eigenvalue of A. We get the following properties:*
a) $V_{\lambda} \subseteq V^{\lambda}$;
b) The subspace V^{λ} is A-invariant.

Proposition 3.4. *Suppose that A is a linear operator in a finite-dimensional vector space V and $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of its characteristic polynomial. If for some vectors $x_1, \ldots, x_k$ such that $x_j \in V^{\lambda_j}$ we have an equality*

$$\sum_{j=1}^{k} x_j = 0 \tag{3.2}$$

then $x_j = 0$, $\forall 1 \leq j \leq k$.

The following two lemmas may help us prove Proposition 3.4.

Lemma 3.5. *Suppose $A \in \mathrm{End}(V)$ and λ is its eigenvalue, then $\forall x \in V^{\lambda}, x \neq 0$, $\exists k \geq 0$ such that $(A - \lambda \mathrm{Id})^k(x)$ is an eigenvector of the linear operator A with an eigenvalue λ.*

Lemma 3.6. *Let v be an eigenvector of a linear operator A with an eigenvalue λ. We claim that for every polynomial $p \in \mathbb{F}[x]$ the element v is an eigenvector of $p(A)$ with the eigenvalue $p(\lambda)$.*

Suppose that $U_1, \ldots, U_k$ are subspaces of a vector space V. Recall that, by definition,

$$\left(V = \bigoplus_{j=1}^{k} U_j\right) \quad \Longleftrightarrow \quad \left(\forall v \in V \exists! (x_1, \ldots, x_k) \text{ such that } x_j \in U_j, \forall 1 \leq j \leq k \text{ and } \sum_{j=1}^{k} x_j = v\right). \tag{3.3}$$

Now we are ready to formulate the main theorem of this lecture.

Proposition 3.7 (Primary Decomposition Theorem)**.** *Suppose that A is a linear operator in a finite-dimensional vector space V and its characteristic polynomial splits:*

$$\chi_A(x) = \prod_{j=1}^{k} (x - \lambda_j)^{m_j}, \tag{3.4}$$

where $\lambda_1, \ldots, \lambda_k$ are distinct, $m_j \geq 1$, $\forall 1 \leq j \leq k$, then we obtain the equality

$$\boxed{V = \bigoplus_{j=1}^{k} V^{\lambda_j};} \tag{3.5}$$

To prove the Primary Decomposition Theorem, one may consider the polynomials

$$q_j := \prod_{1\leq i\leq k,\, i\neq j} (x-\lambda_i)^{m_i} \tag{3.6}$$

and note that

$$\mathrm{im}(q_j(A)) \subseteq V^{\lambda_j}, \quad \forall 1 \leq j \leq k. \tag{3.7}$$

Bézout's Lemma for the $GCD(q_1, \ldots, q_k)$ implies a part of the theorem statement.

Another part of this theorem follows from Proposition 3.4.

Note that one may deduce that $\dim V^{\lambda_j} = m_j$, $\forall 1 \leq j \leq k$ from Schur's Theorem. We are not going to do it here. This fact and other details will follow from the Jordan Canonical Form Theorem that we will prove in the next lecture.

4. Other Useful Characteristics of a Linear Operator

Definition 4.1. Suppose that A is a linear operator in a finite-dimensional vector space V and

$$\chi_A(\lambda) = \lambda^n + \sum_{j=0}^{n-1} b_j \lambda^j. \tag{4.1}$$

The trace $\mathrm{tr}(A)$ of a linear operator A, by definition, is equal to $-b_{n-1}$.

Question 4.2. What can we say about the coefficient b_0?

Note that because the characteristic polynomial of a linear operator does not depend on the choice of the basis, neither does the trace.

Exercise 4.3. Let (a_i^j) be the matrix of a linear operator $A \in \mathrm{End}(V)$ in some basis. Find an expression for $\mathrm{tr}(A)$ in terms of the matrix entries.

Exercise 4.4. Show that

$$\mathrm{tr}(AB) = \mathrm{tr}(BA), \quad \forall A, B \in \mathrm{End}(V). \tag{4.2}$$

Definition 4.5. Suppose that A is a linear operator in a finite-dimensional vector space V. Consider the set

$$I(A) := \{p \in \mathbb{F}[x] \mid p \neq 0, p(A) = 0\}. \tag{4.3}$$

One can show that a minimal degree element of $I(A)$ is determined uniquely up to multiplication by a non-zero scalar. As usual, we may require that its leading coefficient equals 1 to ensure uniqueness. This monic polynomial is called the minimal (annihilating) polynomial for the linear operator A.

Exercise 4.6. Show that the minimal polynomial for a linear operator is indeed well-defined.

Question 4.7. Can the minimal polynomial for some linear operator coincide with its characteristic polynomial?

Chapters from the textbooks relevant to the lecture:

- Friedberg, Insel, Spence, Sections 5.1, 5.2, 7.1;
- Axler, pp. 75-90, 163 - 177;
- Roman, pp. 185-190, 192-197.