LECTURE 10. COMPLEXIFICATION. JORDAN FORM FOR THE REAL CASE.

Please consider proof of every proposition in this lecture to be an exercise.

1. Complexification.

Definition 1.1. Let V be a real vector space. Let us consider the field $\mathbb{C}$ as another vector space over $\mathbb{R}$. The tensor product $\mathbb{C} \otimes V$ is also a real vector space. We will introduce a structure of complex vector space on $\mathbb{C} \otimes V$ by setting

$$\lambda \cdot (z \otimes v) := (\lambda z \otimes v), \quad \forall \lambda, z \in \mathbb{C}, \forall v \in V. \tag{1.1}$$

Here, we define the multiplication by a fixed λ on the set of pairs $\mathbb{C} \times V$. Note that it is bilinear, and then we extend this map to the tensor product $\mathbb{C} \otimes V$ using its universal property. We also make the addition operation the same as in the tensor product $\mathbb{C} \otimes V$. The complex vector space that we have obtained is called the complexification of the real vector space V and is denoted by $V^{\mathbb{C}}$. Also put

$$V_0^{\mathbb{C}} := \{1 \otimes v \mid v \in V\}. \tag{1.2}$$

For every subspace U of the vector space V denote by $U^{\mathbb{C}}$ the following subspace of $V^{\mathbb{C}}$:

$$U^{\mathbb{C}} := \{z \otimes u \mid z \in \mathbb{C}, u \in U\} \tag{1.3}$$

Exercise 1.2. Let V be a real vector space. Show that $\forall v \in V^{\mathbb{C}}$ $\exists u, w \in V$ such that

$$v = 1 \otimes u + i \otimes w. \tag{1.4}$$

Exercise 1.3. Let V be a real vector space. Show that $V^{\mathbb{C}}$ is indeed a complex vector space. Also, prove that if $\dim V < \infty$, then the dimension of $V^{\mathbb{C}}$ over C equals $\dim V$. What can we say about $V_0^{\mathbb{C}}$ and its dimension?

Definition 1.4. Suppose that V is a real vector space and $A \in \mathrm{End}(V)$. We define the complexification $A^{\mathbb{C}} \in \mathrm{End}(V^{\mathbb{C}})$ of this linear operator A by

$$A^{\mathbb{C}}(z \otimes v) := z \otimes (Av). \tag{1.5}$$

Note that

$$\forall v \in V_0^{\mathbb{C}} \quad A^{\mathbb{C}} v \in V_0^{\mathbb{C}}, \tag{1.6}$$

and the restriction of $A^{\mathbb{C}}$ to $V_0^{\mathbb{C}}$ coincides with A. So we can view $A^{\mathbb{C}}$ as an extension of a copy of A defined on $V_0^{\mathbb{C}}$. We are going to exploit this viewpoint a lot in the next section.

Example 1.5. Suppose that V is a real vector space, $\alpha = \{v_1, \ldots, v_n\}$ is a basis of V, $A \in \mathrm{End}(V)$. Then the set $\beta = \{1 \otimes v_1, \ldots, 1 \otimes v_n\}$ is a basis of $V^{\mathbb{C}}$ and the corresponding matrices coincide:

$$[A^{\mathbb{C}}]_\beta = [A]_\alpha. \tag{1.7}$$

Proposition 1.6. *Suppose that V is a real vector space and $A \in \mathrm{End}(V)$. Then we get the following equality for the characteristic polynomials*

$$\chi_{A^{\mathbb{C}}}(x) = \chi_A(x), \quad \forall x \in \mathbb{R}. \tag{1.8}$$

Note that $\chi_A \in \mathbb{R}[x]$. So Proposition 1.6 states that the characteristic polynomial of the linear operator $A^{\mathbb{C}}$ has real coefficients and coincides with χ_A.

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Example 1.7. Suppose that V is a complex vector space, $\gamma = \{v_1, \dots, v_n\}$ is a basis of V, $D \in \mathrm{End}(V)$ and

$$[D]_\gamma = (a_k^l + ib_k^l)_{1 \le k,l \le n} = A + iB, \quad A, B \in \mathrm{Mat}(n \times n, \mathbb{R}). \tag{1.9}$$

Now let us consider V as a vector space over $\mathbb{R}$, denote this vector space by $V^{\mathbb{R}}$ to avoid confusion. Then the map D is a linear operator of $V^{\mathbb{R}}$, denote it by $D^{\mathbb{R}}$. Then $\delta = \{v_1, \dots, v_n, iv_1, \dots, iv_n\}$ is a basis of $V^{\mathbb{R}}$. We get that

$$[D^{\mathbb{R}}]_\delta = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{1.10}$$

Proposition 1.8. *Suppose that V is a real vector space, $\dim V < \infty$. Define* Conj $\in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *by*

$$\mathrm{Conj}(z \otimes v) := \overline{z} \otimes v. \tag{1.11}$$

Then:

a) $\mathrm{Conj}^2 = \mathrm{Id}$*;*

b) $\forall A \in \mathrm{End}(V) \quad \mathrm{Conj} \circ A^{\mathbb{C}} = A^{\mathbb{C}} \circ \mathrm{Conj}$*;*

c) We get that $\frac{1}{2}(\mathrm{Id} + \mathrm{Conj}) \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *and it is a projection operator onto* $V_0^{\mathbb{C}}$*;*

d) We get that $\frac{1}{2i}(Id - \mathrm{Conj}) \in \mathrm{End}((V^{\mathbb{C}})^{\mathbb{R}})$ *and*

$$\mathrm{im}\left(\frac{1}{2i}(Id - \mathrm{Conj})\right) = V_0^{\mathbb{C}}; \tag{1.12}$$

e)

$$\mathrm{im}\left(\frac{1}{2}(\mathrm{Id} + \mathrm{Conj})\right) \oplus \mathrm{im}\left(\frac{1}{2}(Id - \mathrm{Conj})\right) = (V^{\mathbb{C}})^{\mathbb{R}}. \tag{1.13}$$

Proposition 1.9. *Suppose that V is a real finite-dimensional vector space and $A \in \mathrm{End}(V)$. Denote by V^λ a generalized eigenspace of $A^{\mathbb{C}}$. Then* $\mathrm{Conj}(V^\lambda) = V^{\overline{\lambda}}$.

Proposition 1.10. *Suppose that V is a real finite-dimensional vector space and W is a subspace of $V^{\mathbb{C}}$. Then the following conditions are equivalent:*

$$(\mathrm{Conj}(W) = W) \quad \Longleftrightarrow \quad (\textit{There exists a subspace } U \subset V \textit{ such that } W = U^{\mathbb{C}}). \tag{1.14}$$

2. Jordan form for real linear operators.

In this section, we will obtain an analog of Jordan canonical form for the case of real linear operators. We will complexify a real linear operator, get a Jordan basis for its complexification, and then we will show how to obtain from it a Jordan basis for the initial real case.

Proposition 2.1. *Suppose that V is a real vector space, $A \in \mathrm{End}(V)$, $\chi_A(x)$ has no real roots, and*

$$\chi_A(x) = x^2 + cx + d, \quad c, d \in \mathbb{R}. \tag{2.1}$$

Put

$$a := -\frac{c}{2}; \quad b := \frac{\sqrt{4d - c^2}}{2}. \tag{2.2}$$

There exists a basis of V in which the matrix of A has the form

$$R(a, b) := \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{2.3}$$

Proof. Consider

$$\chi_{A^{\mathbb{C}}}(x) = \chi_A(x) = x^2 + cx + d. \tag{2.4}$$

It has the roots $a + bi$ and $a - bi$. Hence, there exists an eigenvector $v \in V^{\mathbb{C}}$ corresponding to the eigenvalue $a + bi$. As we know, there exist $u, w \in V$ such that $v = 1 \otimes u + i \otimes w$. Therefore,

$$A^{\mathbb{C}}(1 \otimes u + i \otimes w) = (a + bi)(1 \otimes u + i \otimes w). \tag{2.5}$$

Expanding this equality, we obtain

$$A(u) = au - bw, \tag{2.6}$$

$$A(w) = bu + aw. \tag{2.7}$$

□

Note that by equality (2.2) $b > 0$.

Exercise 2.2. Show that

$$u = \frac{1}{2}(\mathrm{Id} + \mathrm{Conj})(v); \quad w = \frac{1}{2i}(\mathrm{Id} - \mathrm{Conj})(v). \tag{2.8}$$

Example 2.3. Let $a, b \in \mathbb{R}$. Put $r := \sqrt{a^2 + b^2}$. There exists $\phi \in \mathbb{R}$ such that

$$R(a,b) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} r\cos(\phi) & r\sin(\phi) \\ -r\sin(\phi) & r\cos(\phi) \end{pmatrix} = (r \cdot \mathrm{Id}) \cdot \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}. \tag{2.9}$$

Definition 2.4. Suppose that $a, b \in \mathbb{R}$, $b > 0$. Define a generalization of a Jordan block $J_k(a,b) \in \mathrm{Mat}(2k \times 2k, \mathbb{R})$ as

$$\boxed{J_k(a,b) := \begin{pmatrix} R(a,b) & \mathrm{Id} & 0 & \dots & 0 & 0 \\ 0 & R(a,b) & \mathrm{Id} & \dots & 0 & 0 \\ 0 & 0 & R(a,b) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & R(a,b) & \mathrm{Id} \\ 0 & 0 & 0 & \dots & 0 & R(a,b). \end{pmatrix}} \tag{2.10}$$

Theorem 2.5 (Jordan form for the real case)**.** *Suppose that A is a linear operator in a finite-dimensional real vector space V. There exists a basis α of the vector space V in which the matrix of A has the following block-diagonal form*

$$\boxed{[A]_\alpha = \begin{pmatrix} J_{s_1}(\mu_1) & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\ 0 & J_{s_2}(\mu_2) & \dots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & J_{s_p}(\mu_p) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & J_{k_1}(a_1,b_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & J_{k_2}(a_2,b_2) & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_{k_q}(a_q,b_q) \end{pmatrix}} \tag{2.11}$$

for some, not necessarily distinct,

$$s_1, s_2, \dots, s_p, k_1, k_2, \dots, k_q \in \mathbb{Z}_+,\ \mu_1, \mu_2, \dots, \mu_p, a_1, \dots, a_q \in \mathbb{R},\ b_1, \dots, b_q \in \mathbb{R}_+. \tag{2.12}$$

The sequence of Jordan blocks is determined uniquely up to their permutation.

Proof. Consider $A^{\mathbb{C}} \in \mathrm{End}(V^{\mathbb{C}})$. Then we may represent $V^{\mathbb{C}}$ as a direct sum of the generalized eigenspaces:

$$V^{\mathbb{C}} = \bigoplus_{t=1}^{k} V^{\lambda_t} \tag{2.13}$$

Let us pick an arbitrary λ_t and consider the restriction of $A^{\mathbb{C}}$ to the subspace V^{λ_t}, which we will denote by B.

First, assume that $\lambda_t \in \mathbb{R}$. From Proposition 1.10 we get that V^{λ_t} is invariant under the real operator Conj.

Question 2.6. How to check this assertion?

So there exist $U \subset V$ such that

$$V^{\lambda_t} = U^{\mathbb{C}}. \tag{2.14}$$

The restriction of $A - \lambda_t \mathrm{Id}$ to U is nilpotent.

Question 2.7. Why is this claim true?

Hence, we may construct an ordinary Jordan basis for the restriction of $A - \lambda_t \mathrm{Id}$ to U as we did in the proof in the previous lecture.

Now assume that $\lambda_t \notin \mathbb{R}$, $\lambda_t = a + bi$, $a, b \in \mathbb{R}$. For simplicity's sake, let us first consider the case when the JCF for B consists of a single block $J_r(\lambda_t)$ in a basis $v_1, \dots, v_r$. So we have the following equalities:

$$Bv_1 = \lambda_t v_1, \tag{2.15}$$

$$Bv_2 = \lambda_t v_2 + v_1, \tag{2.16}$$

$$\vdots \qquad \vdots \tag{2.17}$$

$$Bv_r = \lambda_t v_r + v_{r-1}. \tag{2.18}$$

As we know, there exist $u_1, \dots, u_r, w_1, \dots, w_r \in V$ such that

$$v_j = 1 \otimes u_j + i \otimes w_j, \quad \forall 1 \leq j \leq r. \tag{2.19}$$

The linear span over $\mathbb{R}$

$$H := \langle u_1, \dots, u_r, w_1, \dots, w_r \rangle_{\mathbb{R}} \subset V \tag{2.20}$$

is A-invariant.

Exercise 2.8. Show that the matrix of the restriction of A to the subspace H in the basis $u_1, \dots, u_r, w_1, \dots, w_r \in V$ equals $J_r(a.b)$.

Exercise 2.9. Show that

$$V^{\lambda_t} \oplus V^{\overline{\lambda_t}} = (U)^{\mathbb{C}}. \tag{2.21}$$

For completeness, we will briefly describe this construction when the JCF for B does not necessarily consist of one block. Consider a Jordan basis $\{v_{(i|j)} \mid 1 \leq i \leq m_j, 1 \leq j \leq r\}$ for a subspace V^{λ_t}. Suppose it corresponds to the sequence of Jordan blocks $J_{m_1}(\lambda_t), \dots, J_{m_r}(\lambda_t)$. Then $\{\mathrm{Conj}(v_{(1|j)}) \mid 1 \leq j \leq r\}$ is a Jordan basis for the subspace $V^{\overline{\lambda_t}}$. Put

$$S_t := \left\{ \frac{1}{2}(\mathrm{Id} + \mathrm{Conj})(v_{(1|1)}), \frac{1}{2i}(\mathrm{Id} - \mathrm{Conj})(v_{(1|1)}), \dots, \frac{1}{2}(\mathrm{Id} + \mathrm{Conj})(v_{(m_r|r)}), \frac{1}{2i}(\mathrm{Id} - \mathrm{Conj})(v_{(m_r|r)}) \right\} \tag{2.22}$$

Also, define

$$H_t := \langle S_t \rangle. \tag{2.23}$$

H_t is A-invariant.

Exercise 2.10. Show that the matrix of the restriction of A to H_t in the basis S_t has the form

$$\begin{pmatrix} J_{m_1}(a,b) & 0 & 0 & 0 \\ 0 & J_{m_2}(a,b) & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & J_{m_r}(a,b) \end{pmatrix}. \tag{2.24}$$

We are going to use the following claim

Lemma 2.11. *Suppose that $U_1, \dots, U_l$ are subspaces of a real vector space V such that*

$$V^{\mathbb{C}} = \bigoplus_{j=1}^{l} (U_j)^{\mathbb{C}}. \tag{2.25}$$

Then

$$V = \bigoplus_{j=1}^{l} U_j. \tag{2.26}$$

This lemma finishes the proof of the existence of the analog of Jordan basis in the real case.

Now let us prove the uniqueness of this Jordan form. Consider the complexification of the restriction of A to the subspace corresponding to a Jordan block $J_{k_j}(a_j, b_j)$. Its JCF consists of two blocks $J_{k_j}(a_j + b_j i)$ and $J_{k_j}(a_j - b_j i)$. The uniqueness of the JCF in the complex case implies the uniqueness in our case.

Exercise 2.12. Explain this implication.

□

Chapters from the textbooks relevant to the lecture:

- Axler, Ch.9;
- Roman, pp.53-55, 82-83.