LECTURE 13. VARIOUS OPERATORS IN INNER PRODUCT SPACES

Please consider every proposition in this lecture as an exercise.

1. Isometries.

Definition 1.1. An isomorphism A between two inner product spaces V and W over the same field $\mathbb{R}$ or $\mathbb{C}$ is called an isometry if

$$\langle Ax, Ay\rangle_W = \langle x, y\rangle_V, \quad \forall x, y \in V. \tag{1.1}$$

Proposition 1.2. *Suppose that V and W are finite-dimensional inner product spaces over the same field $\mathbb{R}$ or $\mathbb{C}$ and $\dim V = \dim W$. Then there exists an isometry between V and W.*

2. Unitary operators

Definition 2.1. Let V be a complex inner product space. A linear operator A is called unitary if

$$\langle Ax, Ay\rangle = \langle x, y\rangle, \quad \forall x, y \in V. \tag{2.1}$$

Question 2.2. What can we conclude about the invertibility and the inverse of a unitary operator in a finite-dimensional inner product space?

Proposition 2.3. *A is a unitary operator if and only if $AA^* = A^*A = \mathrm{Id}$.*

Proposition 2.4. *Suppose that V is a complex inner product space, $\dim V = n$ and α is an orthonormal basis in V. Then*

$$(A \text{ is unitary}) \iff (\text{The columns(rows) of } [A]_\alpha \text{ form an orthonormal basis in } \mathbb{C}^n). \tag{2.2}$$

The spectral theorem for normal operators implies the following result for unitary operators.

Proposition 2.5. *For every unitary operator A in a finite-dimensional complex inner product space V there exists a basis α in which the matrix of A is diagonal:*

$$[A]_\alpha = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}, \quad \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C},\ |\lambda_j| = 1, 1 \le j \le n. \tag{2.3}$$

The analogs of unitary operators for the real case are called orthogonal.

Definition 2.6. Let V be a real inner product space. A linear operator A is called orthogonal if

$$\langle Ax, Ay\rangle = \langle x, y\rangle, \quad \forall x, y \in V. \tag{2.4}$$

Proposition 2.7. *Suppose that V is a real inner product space, $\dim V = n$ and α is an orthonormal basis in V. Then*

$$(A \text{ is orthogonal}) \iff (\text{The columns(rows) of } [A]_\alpha \text{ form an orthonormal basis in } \mathbb{R}^n). \tag{2.5}$$

Definition 2.8. For $\psi \in \mathbb{R}$ denote by R_ψ the following matrix

$$R_\psi := \begin{pmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{pmatrix}. \tag{2.6}$$

Date: May 3, 2024.

Proposition 2.9. *For every orthogonal operator A in a finite-dimensional real inner product space V there exists a basis α in which*

$$[A]_\alpha = \begin{pmatrix} R_{\psi_1} & 0 & \dots & 0 & 0 & 0 \\ 0 & R_{\psi_2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & R_{\psi_r} & 0 & 0 \\ 0 & 0 & \dots & 0 & \mathrm{Id}_p & 0 \\ 0 & 0 & \dots & 0 & 0 & -\mathrm{Id}_q \end{pmatrix}, \quad p, q, r \geq 0,\ \psi_1, \psi_2, \dots, \psi_r \in \mathbb{R}, \tag{2.7}$$

and by Id_p we denote the identity $p \times p$ matrix. The sizes of zero matrices in the off-diagonal entries vary depending on the sizes of the corresponding blocks on the main diagonal.

Note that $\dim V = 2r + p + q$ in this case.
Also, notice that

$$R_\pi := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{2.8}$$

Suppose that A is orthogonal and $\det(A) = 1$. Recall that $\det(R_\phi) = 1$, $\forall \phi \in \mathbb{R}$. Hence, we conclude that $\dim V$ is even, and we can write all diagonal blocks in the equality (2.7) as 2×2 blocks R_ϕ.

3. Special classes of normal operators

In this section, we will get spectral theorems for four more classes of operators. They can be easily deduced from the spectral theorems for normal operators in complex and real inner product spaces.

Definition 3.1. Let V be a complex inner product space. A linear operator A is called self-adjoint or Hermitian if $A^* = A$.

Proposition 3.2. *For every Hermitian operator A in a finite-dimensional complex inner product space V there exists a basis α in which the matrix is diagonal*

$$[A]_\alpha = \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_n \end{pmatrix}, \quad \mu_1, \mu_2, \dots, \mu_n \in \mathbb{R}. \tag{3.1}$$

Definition 3.3. Let V be a real inner product space. A linear operator A is called symmetric if $A^* = A$.

Proposition 3.4. *For every symmetric operator A in a finite-dimensional real inner product space V there exists a basis α in which the matrix is diagonal*

$$[A]_\alpha = \begin{pmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_n \end{pmatrix}, \quad \mu_1, \mu_2, \dots, \mu_n \in \mathbb{R}. \tag{3.2}$$

Definition 3.5. Let V be a complex inner product space. A linear operator A is called skew-Hermitian if $A^* = -A$.

Proposition 3.6. *For every skew-Hermitian operator A in a finite-dimensional complex inner product space V there exists a basis α in which the matrix is diagonal*

$$[A]_\alpha = \begin{pmatrix} i\mu_1 & 0 & \dots & 0 \\ 0 & i\mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & i\mu_n \end{pmatrix}, \quad \mu_1, \mu_2, \dots, \mu_n \in \mathbb{R}. \tag{3.3}$$

Definition 3.7. Let V be a real inner product space. A linear operator A is called skew-symmetric if $A^* = -A$.

Proposition 3.8. *For every skew-symmetric operator A in a finite-dimensional real inner product space V there exists a basis α in which*

$$[A]_\alpha = \begin{pmatrix} R(0,b_1) & 0 & \dots & 0 & 0 \\ 0 & R(0.b_2) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & R(0.b_r) & 0 \\ 0 & 0 & \dots & 0 & 0_p \end{pmatrix}, \quad p.r \geq 0,\ b_1, b_2, \dots, b_r \in \mathbb{R}, \tag{3.4}$$

and 0_p stands for the zero $p \times p$ matrix.

4. Matrix interpretations of spectral theorems

Definition 4.1. We have introduced several classes of linear operators in inner product spaces - normal, unitary, orthogonal, Hermitian (self-adjoint), symmetric, skew-Hermitian, and skew-symmetric.

For each of those classes we will define a corresponding class of matrices via a similar rule. For instance, if $A \in \mathrm{End}(V)$ is a unitary operator, a basis α is orthonormal then the matrix $[A]_\alpha$ is called unitary. Or, we can state this condition differently:

A matrix $B \in \mathrm{Mat}(n \times n, \mathbb{C})$ is called unitary iff $(\overline{B})^t \cdot B = \mathrm{Id}$.

This definition allows us to reformulate each of the spectral theorems in terms of the matrices. We will write one example.

Proposition 4.2. *Let $A \in \mathrm{Mat}(n \times n, \mathbb{C})$ be a normal complex matrix. Then there exist unitary matrix U and a diagonal matrix D such that*

$$A = UDU^{-1}. \tag{4.1}$$

5. Positive Operators

Definition 5.1. A self-adjoint (or symmetric) operator A in an inner product space V over $\mathbb{C}$ (or, over $\mathbb{R}$) is called

a) non-negative (positive semi-definite if

$$\langle Ax, x\rangle \geq 0, \quad \forall x \in V; \tag{5.1}$$

b) positive definite if

$$\langle Ax, x\rangle > 0, \quad \forall x \neq 0. \tag{5.2}$$

Proposition 5.2. *A self-adjoint (or symmetric) operator P in an inner product space V over $\mathbb{C}$ (or, over $\mathbb{R}$) is non-negative iff its every eigenvalue is non-negative. A self-adjoint (or symmetric) operator P in an inner product space V over $\mathbb{C}$ (or, over $\mathbb{R}$) is positive definite iff its every eigenvalue is positive.*

Proposition 5.3. *A self-adjoint operator P in an inner product space V over $\mathbb{C}$ is positive definite iff a new map $B : V \times V \to \mathbb{C}$ defined as*

$$B(x, y) := \langle x, Py \rangle \tag{5.3}$$

also determines an inner product.

Proposition 5.4. *Suppose that A is a self-adjoint (or symmetric) operator in an inner product space V over $\mathbb{C}$ (or over $\mathbb{R}$). Then*

a)

$$(A \text{ is positive}) \quad \Longleftrightarrow \quad (A = C^*C \text{ for } C \in \mathrm{End}(V)); \tag{5.4}$$

b)

$$(A \text{ is positive definite}) \quad \Longleftrightarrow \quad (A = C^*C \text{ for } C \in \mathrm{GL}(V)). \tag{5.5}$$

Proposition 5.5 (Sylvester's Criterion)**.** *Suppose that A is a self-adjoint (or symmetric) operator in an inner product space V over $\mathbb{C}$ (or, over $\mathbb{R}$), and $[A]_\alpha$ is its matrix in some orthonormal basis α. Then*

$$\begin{aligned}(A \text{ is positive definite}) \iff \\ \Longleftrightarrow (\text{The minors constructed from the rows } 1, 2, \ldots, k \text{ and the columns } 1, 2, \ldots, k \text{ of } [A]_\alpha \\ \text{are positive for every } 1 \leq k \leq n).\end{aligned} \tag{5.6}$$

6. Polar Decomposition

Proposition 6.1 (Polar Decomposition(complex case)**.** *Let V be an inner product space over $\mathbb{C}$. For every $A \in \mathrm{End}(V)$ there exist a non-negative operator P with a unitary operator U such that $A = PU$. If A is invertible, then P and U are uniquely determined.*

Proposition 6.2 (Polar Decomposition(real case))**.** *Let V be an inner product space over $\mathbb{R}$. For every $A \in \mathrm{End}(V)$ there exist a non-negative operator P with an orthogonal operator U such that $A = PU$. If A is invertible, then P and U are uniquely determined.*

Chapters from the textbooks relevant to the lecture:

- Friedberg, Insel, Spence, Sections 6.5, 6.6;
- Axler, pp.144-158;
- Roman, pp. 238-254.