LECTURE 14. BILINEAR FORMS.

Please treat every proposition in this lecture as an exercise.

1. Positive Operators

Definition 1.1. A self-adjoint (or symmetric) operator P in an inner product space V over $\mathbb{C}$ (or, over $\mathbb{R}$) is called

a) non-negative (positive semi-definite) if

$$\langle Px, x\rangle \geq 0, \quad \forall x \in V; \tag{1.1}$$

b) positive definite if

$$\langle Px, x\rangle > 0, \quad \forall x \neq 0. \tag{1.2}$$

We can motivate our definition by the following statement.

Proposition 1.2. *A self-adjoint operator P in an inner product space V over $\mathbb{C}$ is positive definite iff a new map $B : V \times V \to \mathbb{C}$ defined as*

$$B(x, y) := \langle Px, y\rangle \tag{1.3}$$

also determines an inner product.

Proposition 1.3. *A self-adjoint (or symmetric) operator P in an inner product space V over $\mathbb{C}$ (or, over $\mathbb{R}$) is non-negative iff its every eigenvalue is non-negative. A self-adjoint (or symmetric) operator P in an inner product space V over $\mathbb{C}$ (or, over $\mathbb{R}$) is positive definite iff its every eigenvalue is positive.*

Proposition 1.4. *Suppose that A is a self-adjoint (or symmetric) operator in an inner product space V over $\mathbb{C}$ (or over $\mathbb{R}$). Then*

a)

$$(A \text{ is positive}) \iff (A = C^*C \text{ for } C \in \mathrm{End}(V)); \tag{1.4}$$

b)

$$(A \text{ is positive definite}) \iff (A = C^*C \text{ for } C \in \mathrm{GL}(V)). \tag{1.5}$$

Proposition 1.5 (Sylvester's Criterion)**.** *Suppose that A is a self-adjoint (or symmetric) operator in an inner product space V over $\mathbb{C}$ (or, over $\mathbb{R}$), and $[A]_\alpha$ is its matrix in some orthonormal basis α. Then*

$$\begin{aligned}(A \text{ is positive definite}) \iff \\ \iff (\text{The minors constructed from the rows } 1, 2, \dots, k \text{ and the columns } 1, 2, \dots, k \text{ of } [A]_\alpha \\ \text{are positive for every } 1 \leq k \leq n).\end{aligned} \tag{1.6}$$

Date: May 10, 2024.

2. Polar Decomposition

Lemma 2.1. *Suppose A is a linear operator in an inner product space V. Then there exists a unique non-negative operator B such that $B^2 = A * A$.*

Lemma 2.2. *A linear operator U in an inner product space V is unitary (orthogonal) if and only if for every $x \in V$ we get the equality $|Ux| = |x|$.*

Those lemmas allow us to obtain the main results of this section.

Proposition 2.3 (Polar Decomposition(complex case)**.** *Let V be an inner product space over $\mathbb{C}$. For every $A \in \mathrm{End}(V)$ there exist a non-negative operator P with a unitary operator U such that $A = UP$. If A is invertible, then P and U are uniquely determined.*

Proposition 2.4 (Polar Decomposition(real case))**.** *Let V be an inner product space over $\mathbb{R}$. For every $A \in \mathrm{End}(V)$ there exist a non-negative operator P with an orthogonal operator U such that $A = UP$. If A is invertible, then P and U are uniquely determined.*

Question 2.5. Would the formula of polar decomposition still exist if we make the positive operator the first multiple in the product?

3. Singular Value Decomposition and the Moore-Penrose pseudoinverse

In this section $\mathbb{F}$ stands for $\mathbb{R}$ or $\mathbb{C}$.

Proposition 3.1. *For an arbitary matrix $A \in \mathrm{Mat}(m \times n, \mathbb{F})$ there exist unitary matrices $U_1 \in \mathrm{Mat}(m \times m, \mathbb{F})$, $U_1 \in \mathrm{Mat}(n \times n, \mathbb{F})$ and a diagonal matrix $D \in \mathrm{Mat}(m \times n, \mathbb{F})$ with non-negative values on the main diagonal such that*

$$A = U_1 D U_2 \tag{3.1}$$

Question 3.2. Is the singular value decomposition (3.1) uniquely determined by the matrix A?

Exercise 3.3. Formulate and prove an analog of the SVD for the real case.

Definition 3.4. Suppose that $A = U_1 D U_2$ is a singular value decomposition for the matrix $A \in \mathrm{Mat}(m \times n, \mathbb{F})$, where the diagonal matrix $D = (d_i^j) in \mathrm{Mat}(m \times n, \mathbb{F})$. Define a diagonal matrix $D^+ = (t_i^j) \in \mathrm{Mat}(n \times m, \mathbb{F})$ by

$$t_i^i = \begin{cases} \frac{1}{d_i^i}, & \text{if } d_i^i \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

The Moore-Penrose pseudoinverse $A^+ \in \mathrm{Mat}(n \times m, \mathbb{F})$ of the matrix A is defined by the formula

$$A^+ = U_2^* D^+ U_1^*. \tag{3.3}$$

Exercise 3.5. Show that A^+ does not depend on the choice of the SVD for the matrix A.

Proposition 3.6. *If A is invertible then A^+ coincides with A^{-1}.*

Proposition 3.7. *Suppose that $A \in \mathrm{Mat}(m \times n, \mathbb{F})$, $y \in \mathbb{F}^m$. Then $A^+ y \in \mathbb{F}^n$ has the minimal norm among the vectors $x \in \mathbb{F}^n$ satisfying the condition that*

$$|Ax - y| \leq \inf_{z \in \mathbb{F}^n} |Az - y|. \tag{3.4}$$

4. Bilinear forms

Definition 4.1. Let V be a vector space over $\mathbb{F}$. A map $B : V \times V \to \mathbb{F}$ is called a bilinear form if

$$B(ax + by, z) = aB(x, z) + bB(y, z); \tag{4.1}$$

$$B(x, ay + bz) = aB(x, y) + bB(x, z). \tag{4.2}$$

A bilinear form B is called symmetric if

$$B(x, y) = B(y, x). \tag{4.3}$$

A bilinear form B is called skew-symmetric if

$$B(x, y) = -B(y, x). \tag{4.4}$$

Proposition 4.2. *Suppose that B is a bilinear form in a vector space V and $\alpha = \{v_1, v_2, \ldots, v_n\}$ is a basis of V. Define the matrix of a bilinear form B in the basis α as*

$$\left(B(v_i, v_j)\right)_{1 \le i,j \le n}. \tag{4.5}$$

The bilinear form is completely determined by its matrix in the basis α. This correspondence establishes an isomorphism between the space of matrices $\mathrm{Mat}(n \times n, \mathbb{F})$ and the space of bilinear forms in V.

Remark 4.3. Suppose that the matrix of a bilinear form B in a basis α equals D, x and y are arbitrary vectors, $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are their coordinates in the basis α. Then

$$B(x, y) = (x_1, \ldots, x_n) D \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}. \tag{4.6}$$

Definition 4.4. Two matrices $A, B \in \mathrm{Mat}(n \times n, \mathbb{F})$ are called congruent if there exists an invertible matrix C such that

$$A = C^t B C. \tag{4.7}$$

We may apply some of the results for symmetric and skew-symmetric operators from the previous lecture to bilinear forms in a real inner product space. Note that for an orthogonal operator C, we have that $C^t B C = C^{-1} B C$.

Proposition 4.5. *For every symmetric form in a finite-dimensional inner product space over $\mathbb{R}$ there exists an orthonormal basis in which its matrix is diagonal, and its diagonal entries are determined uniquely, up to permutation, by the form.*

Proposition 4.6. *For every skew-symmetric form in a finite-dimensional inner product space over $\mathbb{R}$ there exists an orthonormal basis in which its matrix has the form*

$$\begin{pmatrix} R(0, b_1) & 0 & \ldots & 0 & 0 \\ 0 & R(0.b_2) & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & R(0.b_r) & 0 \\ 0 & 0 & \ldots & 0 & 0_p \end{pmatrix}. \tag{4.8}$$

The numbers $r, p, b_1, \ldots, b_r$ are determined uniquely, up to permutation, by the form.

Next, we will consider the case of a general vector space, possibly without a defined inner product.

Lemma 4.7. *Suppose that B is a symmetric bilinear form in a finite-dimensional inner product space V over $\mathbb{R}$ or over $\mathbb{C}$ and $B(v,v) \neq 0$ for some $v \in V$. Then*

$$V = \operatorname{span}\{v\} \oplus \{x \in V | B(x,v) = 0\}. \tag{4.9}$$

Proposition 4.8 (Sylvester's Law of Inertia)**.** *For every symmetric form in a finite-dimensional vector space over $\mathbb{R}$ there exists a basis in which its matrix has the form*

$$\begin{pmatrix} \mathrm{Id}_k & 0 & 0 \\ 0 & -\mathrm{Id}_l & 0 \\ 0 & 0 & 0_m \end{pmatrix}. \tag{4.10}$$

The numbers k, l, and m are uniquely determined by the bilinear form.

For the existence part, one may use the lemma. For the uniqueness part, we may consider three subspaces spanned by the basis vectors corresponding to the blocks Id_k, Id_l, and 0_m. Note that the third subspace may be also defined as

$$\operatorname{rad}(B) := \{x \in V | b(x,y) = 0, \, \forall y \in V\}. \tag{4.11}$$

Proposition 4.9. *For every symmetric form in a finite-dimensional vector space over $\mathbb{C}$ there exists a basis in which its matrix has the form*

$$\begin{pmatrix} \mathrm{Id}_k & 0 \\ 0 & 0_l \end{pmatrix}. \tag{4.12}$$

The numbers k and l are uniquely determined by the bilinear form.

Proposition 4.10. *For every skew-symmetric form in a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ there exists a basis in which its matrix has the form*

$$\begin{pmatrix} 0 & \mathrm{Id}_k & 0 \\ -\mathrm{Id}_k & 0 & 0 \\ 0 & 0 & 0_l \end{pmatrix}. \tag{4.13}$$

The numbers k and l are uniquely determined by the bilinear form.

Here, we may first establish an analog of Lemma 4.7.

Chapters from the textbooks relevant to the lecture:

- Friedberg, Insel, Spence, Sections 6.7;
- Axler, pp.144-154;
- Roman, pp. 250-253, 259-264, 443-449.