LECTURE 15. HILBERT SPACES

Please consider every proposition in this lecture as an exercise. Note that the vector spaces in this lecture are not necessarily finite-dimensional.

1. Definitions.

If we do not restrict ourselves to finite-dimensional vector spaces only, it is natural to distinguish a class of Hilbert spaces among all inner product spaces. To introduce it, we will begin with several topological definitions.

Definition 1.1. For a sequence $(x_n)_{n\geq 1}$ of elements of a normed space V we say that (x_n) converges to $v \in V$ if

$$\lim_{n\to\infty} |x_n - v| = 0. \tag{1.1}$$

We also write

$$\lim_{n\to\infty} x_n = v \tag{1.2}$$

in this case.

Definition 1.2. Suppose that V is a normed space. A sequence $(x_n)_{n\geq 1}$ is called a Cauchy sequence if $\forall \varepsilon > 0\, \exists n \in \mathbb{Z}_+$ such that the inequality $|x_k - x_m| < \varepsilon$ holds for all integers $k, m \geq n$.

Definition 1.3. A normed space V is called complete if for every Cauchy sequence $(x_n)_{n\geq 1}$ of elements of V there exists $v \in V$ such that (x_n) converges to v.

Question 1.4. Is it true that every converging sequence is a Cauchy sequence?

Definition 1.5. A complete inner product space over $\mathbb{R}$ or $\mathbb{C}$ is called a Hilbert space.

Question 1.6. Is a finite-dimensional inner product space a Hilbert space?

Exercise 1.7. Consider the following set of complex sequences

$$l^2 := \{(z_n)_{n\geq 1} \mid \sum_{n=1}^{\infty} |z_n|^2 < \infty\}. \tag{1.3}$$

Show that it is a vector space with respect to the termwise addition and multiplication by scalars. Also, show that l^2 is an inner product space if we introduce the inner product as

$$\langle (z_n), (t_n) \rangle := \sum_{n=1}^{\infty} z_n \overline{t_n}. \tag{1.4}$$

Show that l_2 is a Hilbert space.

Exercise 1.8. Consider the following set of complex functions

$$L^2(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{C} \mid \int_{-\infty}^{+\infty} |f(x)|^2\, dx < \infty\}. \tag{1.5}$$

Show that it is a vector space with respect to the pointwise addition and multiplication by scalars. Also, show that $L^2(\mathbb{R})$ is an inner product space if we introduce the inner product as

$$\langle f, g \rangle := \int_{-\infty}^{+\infty} f(x)\overline{g(x)}\, dx. \tag{1.6}$$

Show that $L^2(\mathbb{R})$ is a Hilbert space.

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Definition 1.9. Suppose that M is a subset of a normed space V. An element $v \in V$ is called a limit point of M if $\forall \varepsilon > 0$ we have that

$$\{x \in V \mid 0 < |x - v| < \varepsilon\} \cap M \neq \{\emptyset\}. \tag{1.7}$$

Denote the set of limit points of the set M by M'. The set M is called closed if $M' \subseteq M$. The set

$$\overline{M} := M \cup M' \tag{1.8}$$

is called the closure of M. A set M is said to be dense in V if $\overline{M} = V$.

Proposition 1.10. *For every inner product space V there exists a Hilbert space H and a monomorphism $A \in \mathrm{Hom}(V, H)$ such that A preserves the inner product and $\mathrm{im}(A)$ is dense in H.*

To prove this proposition, one may construct a completion of a given inner product space. We can consider a vector space of Cauchy sequences and define an equivalence relation for two sequences having the same limit.

Proposition 1.11. *A subspace of a Hilbert space is also a Hilbert space iff it is closed.*

It is sufficient to check the completeness of a subspace to prove this claim.

Question 1.12. Is an arbitrary finite-dimensional subspace of a Hilbert space closed?

2. Orthogonal Complement.

Definition 2.1. Let V be an inner product space. We say that a series

$$\sum_{j=1}^{\infty} x_j, \quad x_j \in V, \tag{2.1}$$

converges to a vector $v \in V$ if

$$\lim_{n\to\infty} \sum_{j=1}^{n} x_j = v. \tag{2.2}$$

We say that a series

$$\sum_{j=1}^{\infty} x_j, \quad x_j \in V, \tag{2.3}$$

is absolutely converging if the series

$$\sum_{j=1}^{\infty} |x_j| \tag{2.4}$$

converges.

Exercise 2.2. Show that an inner product V is complete iff every absolutely converging series in V converges.

Proposition 2.3. *Suppose that V is an inner product space and U is its complete subspace. Then for every $v \in V$ there exists a unique element $u_0 \in U$ such that $\langle v - u_0, u\rangle = 0$, $\forall u \in U$. This u_0 is the element in which the following infimum is attained*

$$\inf_{u\in U} |v - u|. \tag{2.5}$$

We may begin the proof by showing that this infimum is attained at some point of U. After that, we can deduce that this point satisfies the orthogonality condition.

This proposition allows us to obtain an analog of finite-dimensional decomposition into the direct sum for the case of Hilbert spaces.

Proposition 2.4. *Let U be a closed subspace of a Hilbert space H. We have the following representation*

$$U \oplus U^{\perp} = H, \tag{2.6}$$

where, as usual,

$$U^{\perp} := \{y \in V \mid \langle y, u\rangle = 0,\ \forall u \in U\}. \tag{2.7}$$

Exercise 2.5. Suppose that U is a closed subspace of a Hilbert space H, and W_1, W_2 are also subspaces of H such that

$$W_1 \oplus U = H = W_2 \oplus U, \quad W_1, W_2 \subseteq U^{\perp}. \tag{2.8}$$

Is it possible that $W_1 \neq W_2$?

Proposition 2.6. *Let U be a subspace of a Hilbert space H. Then*

$$(U^{\perp})^{\perp} = \overline{U}. \tag{2.9}$$

Proposition 2.7. *Let S be a subset of a Hilbert space H. Then* $\operatorname{span}(S)$ *is dense in H iff $S^{\perp} = \{0\}$.*

3. Hilbert Basis

The role of an infinite orthonormal basis is slightly different from what we saw in the case of finite-dimensional inner product spaces.

Exercise 3.1. Suppose that an inner product space V has an infinite orthonormal basis. Can V be complete?

We will introduce a replacement for the notion of an orthonormal basis in the case of Hilbert spaces.

Definition 3.2. A maximal orthonormal set in a Hilbert space H is called a Hilbert basis of H.

Proposition 3.3. *For every orthonormal set of vectors α in a Hilbert space H there exists a Hilbert basis β that contains α.*

Here we may need to use Zorn's Lemma.

Definition 3.4. A Hilbert space is called separable if it has a countable Hilbert basis.

Proposition 3.5. *Suppose that $\{e_1, e_2, \dots\}$ is a countable orthonormal set in a Hilbert space H. Then for every $x \in H$ the series*

$$\sum_{j=1}^{\infty} \langle x, e_j\rangle e_j \tag{3.1}$$

converges absolutely.

Definition 3.6. Suppose that $\{e_1, e_2, \dots\}$ is a countable orthonormal set in a Hilbert space H. For every $x \in H$ put

$$\widehat{x} = \sum_{j=1}^{\infty} \langle x, e_j\rangle e_j \tag{3.2}$$

Proposition 3.7. *Suppose that $S = \{e_1, e_2, \dots\}$ is a countable orthonormal set in a Hilbert space H. Then we get the following results about $\widehat{x}$:*

a)

$$|x - \widehat{x}| = \inf_{y \in \overline{\text{span}(S)}} |x - y|; \tag{3.3}$$

b) (Bessel's inequality) $|\widehat{x}| \leq |x|, \quad \forall x \in H;$

c)

$$(|\widehat{x}| = |x|) \quad \Longleftrightarrow \quad (x \in \overline{\text{span}(S)}). \tag{3.4}$$

Proposition 3.8 (Parseval's identity)**.** *Suppose that $\alpha = \{e_1, e_2, \dots\}$ is a countable orthonormal basis in a Hilbert space H. Then we have the following identity*

$$\langle x, y \rangle = \sum_{j=1}^{\infty} \langle x, e_j \rangle \overline{\langle y, e_j \rangle}, \quad \forall x, y \in H. \tag{3.5}$$

We may first establish that the inner product is continuous in each variable. Afterward, using part c) of the previous proposition, we can prove the identity by approximating x and y.

Proposition 3.9. *For every separable complex (or real) Hilbert space H there exists an isometry between H and $l^2(\mathbb{C})$ (or $l^2(\mathbb{R})$).*

Definition 3.10. Suppose that H_1 and H_2 are Hilbert spaces over the same field, A is a linear map from H_1 to H_2. The map A is called bounded if the set

$$\{\frac{|Ax|}{|x|} \mid x \in H_1, x \neq 0\} \tag{3.6}$$

bounded. Then the supremum of this set is called the norm of A.

Proposition 3.11. *Suppose that H_1 and H_2 are Hilbert spaces over the same field, A is a linear map from H_1 to H_2. Then*

$$(A \text{ is bounded}) \quad \Longleftrightarrow \quad (A \text{ is continuous at } x \text{ for every } x \in H_1). \tag{3.7}$$

Definition 3.12. Let H be a Hilbert space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Then the set of all bounded $f \in \text{Hom}(H, \mathbb{F})$ is called the continuous dual space of H and is denoted by H^*.

Proposition 3.13 (The Riesz Representation Theorem)**.** *Let H be a Hilbert space. For every $f \in H^*$ there exists a unique $z \in H$ such that*

$$f(x) = \langle x, z \rangle, \quad \forall x \in H. \tag{3.8}$$

We may prove this statement by showing that for every non-zero $f \in H^*$ the subspace $(\ker(f))^{\perp}$ is one-dimensional.

Chapters from the textbooks relevant to the lecture:

- Friedberg, Insel, Spence, Section 1.7;
- Roman, pp. 48-51, 325-352.