PROBLEM SET 1, DUE FEBRUARY 11

1. Vector Spaces

Problem 1.1. *Which of the following sets are subspaces of the space of polynomials with real coefficients* $\mathbb{R}[x]$*:*

(1) $\{p \in \mathbb{R}[x] \mid p(1) = 0\}$*;*
(2) $\{p \in \mathbb{R}[x] \mid p(1) = 1\}$*;*
(3) $\{p \in \mathbb{R}[x] \mid p(0) = 0, p(1) = 0\}$*;*
(4) $\{p \in \mathbb{R}[x] \mid p(0)p(1) = 0\}$*?*

Problem 1.2. *Consider the operation of addition on the set of subspaces of a fixed vector space* V*.*

(1) *Is the set of subspaces a monoid?*
(2) *Is it true that for any* $U_1, U_2, U_3 \subseteq V$ *the following equality holds*

$$(U_1 + U_2) \cap U_3 = U_1 \cap U_3 + U_2 \cap U_3?$$

Problem 1.3. *Denote by* L *the vector space of all maps from* $\mathbb{R}$ *to itself. Consider its subspaces* $L_{even} := \{f : \mathbb{R} \to \mathbb{R} \mid f(-x) = f(x), \forall x \in \mathbb{R}\}$ *and* $L_{odd} := \{f : \mathbb{R} \to \mathbb{R} \mid f(-x) = -f(x), \forall x \in \mathbb{R}\}$*. Is it true that* $L = L_{even} \oplus L_{odd}$*?*

Problem 1.4. *Consider*

$$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mid \sum_{j=1}^{n} j x_j = 0 \right\} \subset \mathbb{R}^n.$$

Does there exist a subspace W *such that* $\mathbb{R}^n = U \oplus W$*?*

Problem 1.5. *Let* $\mathbb{R}[x]$ *be the space of polynomials with real coefficients and consider its subspace*

$$U = \{p \in \mathbb{R}[x] \mid p(1) = 0\}.$$

Does there exist a subspace W *such that* $\mathbb{R}[x] = U \oplus W$*?*

Problem 1.6. *Suppose that* U_1*,* U_2 *and* U_3 *are subspaces of a vector space* V*. Is it true that* $V = U_1 \oplus U_2 \oplus U_3$ *if and only if the following conditions hold:*

(1) $V = U_1 + U_2 + U_3$*;*
(2) $U_1 \cap U_2 = \{0\}$*;*
(3) $U_1 \cap U_3 = \{0\}$*;*
(4) $U_2 \cap U_3 = \{0\}$*?*

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