PROBLEM SET 2, DUE FEBRUARY 18

2. Basis, dimension

Problem 2.1. *Find a basis for the subspace*

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \mid \sum_{j=1}^{n} x_j = 0 \right\} \subset \mathbb{R}^n.$$

Problem 2.2. *Consider the vector space*

$$\mathbb{R}_{n-1}[x] := \{ \sum_{j=0}^{n-1} a_j x^j \mid a_0, \dots, a_{n-1} \in \mathbb{R} \} \tag{2.1}$$

of polynomials of degree not greater than $n-1$. Assume that $x_1, \dots, x_n$ are n pairwise distinct real numbers. Put

$$q_j(x) := \prod_{1 \le i \le n,\, i \ne j} \frac{x - x_i}{x_j - x_i}. \tag{2.2}$$

Show that $\{q_1, \dots, q_n\}$ is a basis of $\mathbb{R}_{n-1}[x]$.

Problem 2.3. *Find a basis for the subspace*

$$\{ p \in \mathbb{R}_n[x] \mid p(0) = 0, p(1) = 0 \} \subset \mathbb{R}_n[x],$$

where $\mathbb{R}_n[x]$ is the space of polynomials with real coefficients of degree not greater than n.

Problem 2.4. *a) Does there exist a basis $\{p_1, p_2, p_3, p_4\}$ of $\mathbb{R}_3[x]$ such that none of the polynomials p_1, p_2, p_3, p_4 has degree 3?*

b) Does there exist a basis $\{p_1, p_2, p_3, p_4\}$ of $\mathbb{R}_3[x]$ such that none of the polynomials p_1, p_2, p_3, p_4 has degree 2?

Problem 2.5. *Denote by $\mathrm{Mat}(n \times n, \mathbb{R})$ the vector space of real-valued $n \times n$ matrices, and by U the subspace of symmetric matrices.*

a) Find a basis of U.

b) Find a subspace W such that

$$\mathrm{Mat}(n \times n, \mathbb{R}) = U \oplus W. \tag{2.3}$$

Problem 2.6. *Assume that U and W are subspaces of a vector space V, and*

$$\dim U = k \le dim W = l \le \dim V = n \tag{2.4}$$

What are possible values of $\dim(U \cap W)$? In this problem we may take the result of Proposition 3.8 of Lecture 2 for granted.

Date: February 16, 2023.