PROBLEM SET 4, DUE MARCH 3

4. Change of Basis. Dual spaces.

Problem 4.1. *Let A be $m \times n$ matrix. The linear span of its rows is a subspace of the vector space $\mathbb{F}^n$, the linear span of its columns is a subspace of the vector space $\mathbb{F}^m$. Show that the dimensions of those two subspaces are equal.*

Problem 4.2. *Suppose that V, U, W are finite-dimensional vector spaces, $A \in \mathrm{Hom}(V, U)$, $B \in \mathrm{Hom}(U, W)$. Show that*

$$\dim(\ker(BA)) \leq \dim(\ker(A)) + \dim(\ker(B)).$$

Problem 4.3. *Let V be a finite-dimensional space over a field $\mathbb{F}$. Assume that $A \in End(V)$ such that*

$$AB = BA, \quad \forall B \in \mathrm{End}(V).$$

Does there always exist $\lambda \in \mathbb{F}$ such that $A = \lambda \mathrm{Id}$?

Problem 4.4. *Suppose that P is a linear operator in a vector space V. Show that $P^2 = P$ if and only if the following two conditions hold:*

(1) $V = \ker P \oplus \mathrm{im} P$;
(2) $Px = x, \quad \forall x \in \mathrm{im} P$.

P is called a projection operator.

Problem 4.5. *Let U be a subspace of a finite-dimensional space V. Define the annhilator of U by*

$$\mathrm{ann}(U) := \{f \in V^* \mid f(u) = 0, \forall u \in U\}.$$

Show that $\mathrm{ann}(U)$ is a subspace of V^*. Express $\dim(\mathrm{ann}(U))$ using $\dim V$ and $dim U$.

Problem 4.6. *Suppose that A is a linear operator in a finite-dimensional vector space V. Show that*

$$\ker(A^*) = \mathrm{ann}(\mathrm{im}(A)).$$

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