PROBLEM SET 5-6, DUE MARCH 15

6. Tensor Products of Vector Spaces. Determinant.

Problem 6.1. *Show that every permutation $\rho \in S_n$ can be represented as some product of the permutations $(1,2)$, $(2,3), \ldots, (n-1,n)$.*

Problem 6.2. *Suppose that V and U are finite-dimensional vector spaces over a field $\mathbb{F}$, a set $\{v_1, \ldots, v_n\}$ is a basis of V, and a set $\{u_1, \ldots, u_m\}$ is a basis of U. For an arbitrary element $x \in V \otimes U$ we write the corresponding linear combination*

$$x = \sum_{\substack{1 \le i \le n \\ 1 \le j \le m}} a_{ij}(x) v_i \otimes u_j, \quad a_{ij}(x) \in \mathbb{F}.$$

The rank of the element x is, by definition, the rank of the matrix

$$(a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le m}}.$$

Show that the rank of the element x does not depend on the choice of the bases in V and U.

Problem 6.3. *Does there exist an element of $S^2(V)$ of rank one? Does there exist an element of $\Lambda^2(V)$ of rank one?*

Problem 6.4. *Suppose that* $\dim V = n$ *and* $\mathrm{char}(\mathbb{F}) \neq 2$. *Find* $\dim S^k(V)$.

Problem 6.5. *Assume that* $\mathrm{char}(\mathbb{F}) \neq 2$. *Is it always true that*

$$V^{\otimes 2} = S^2(V) \oplus \Lambda^2(V) \quad ?$$

Problem 6.6. *Suppose that* $\dim V = n$, $\mathrm{char}(\mathbb{F}) \neq 2$ *and* $\alpha \in \Lambda^n(V) \neq 0$. *Define the map $A_\alpha : \Lambda^{n-1}(V) \to V^*$ by the following rule. For an arbitrary $\beta \in \Lambda^{n-1}(V), x \in V$ their wedge product $\beta \wedge x$ belongs to $\Lambda^n(V)$, Therefore, there exists $c(\alpha, \beta, x) \in \mathbb{F}$ such that*

$$\beta \wedge x = c(\alpha, \beta, x) \cdot \alpha.$$

By definition, put

$$(A_\alpha(\beta))(x) := c(\alpha, \beta, x).$$

Show that A_α is well-defined and $A_\alpha \in \mathrm{Hom}(\Lambda^{n-1}(V), V^*)$.

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