PROBLEM SET 7, DUE MARCH 24

7. Polynomials. Caley-Hamilton Theorem.

Problem 7.1. *Suppose that polynomials* $p_1, \dots, p_n \in \mathbb{F}[x]$ *satisfy the condition*

$$GCD(p_i, p_j) = 1, \quad \forall i \neq j. \tag{7.1}$$

Consider the polynomials $q_1, \dots, q_n$ *defined by*

$$q_i := \prod_{j \neq i} p_j, \quad 1 \leq i \leq n. \tag{7.2}$$

Is it always true that

$$GCD(q_1, \dots, q_n) = 1? \tag{7.3}$$

Problem 7.2. *Does there exist a field* $\mathbb{F}$ *and an irreducible polynomial* $p \in \mathbb{F}[x]$ *such that* $\deg(p) > 2$*?*

Problem 7.3. *Suppose that we know all coefficients of the characteristic polynomial of a linear operator* $A \in \mathrm{GL}(V)$*. Find the coefficients of the characteristic polynomial of* A^{-1}*.*

Problem 7.4. *Does there exist a vector space* V*,* $\dim V > 1$*, a linear operator* $A \in \mathrm{End}(V)$ *such that there are only two* A*-invariant subspaces -* $\{0\}$ *and* V*?*

Problem 7.5. *Suppose that* v *is a vector space over a field* $\mathbb{F}$*,* $\lambda \in \mathbb{F}$*,* $A, B \in \mathrm{End}(V)$*,* $AB = BA$*, and*

$$U = \{x \in V \mid Ax = \lambda x\}.$$

Does U *have to be* B*-invariant?*

Problem 7.6. *What is a minimal number* k *such that there exist* $A_1, A_2, \dots, A_k \in \mathrm{End}(\mathbb{C}^n)$ *which products span the vector space* $\mathrm{End}(\mathbb{C}^n)$*?*

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