PROBLEM SET 8, DUE MARCH 31

8. Eigenvalues and Eigenvectors.

Problem 8.1. *For which positive integers n there exists a linear operator $A \in \mathrm{End}(\mathbb{R}^n)$ such that it has no eigenvalues?*

Problem 8.2. *A linear operator $A \in \mathrm{End}(V)$ is called an involution if $A^2 = \mathrm{Id}$. Construct a bijection between the set of involutions in a finite-dimensional vector space V and the set of projection operators in the same vector space.*

Problem 8.3. *Let V be a finite-dimensional vector space. Show that*

$$\mathrm{Spec}(AB) = \mathrm{Spec}(BA),\ \forall A, B \in \mathrm{End}(V).$$

Problem 8.4. *Suppose that A is a linear operator in a finite-dimensional vector space V and $\mathrm{rk}(A) = k$. Show that the characteristic polynomial of A has at most $k+1$ distinct roots.*

Problem 8.5. *Formulate and prove a criterion of invertibility of a square matrix in terms of the set of the roots of the characteristic polynomial.*

Problem 8.6. *Let V be a finite-dimensional vector space. Suppose that $f \in (\mathrm{End}(V))^*$ does not depend on the choice of the basis, i.e.*

$$f(CAC^{-1}) = f(A), \quad \forall A \in \mathrm{End}(V),\ \forall C \in \mathrm{GL}(V).$$

Show that $\exists c \in \mathbb{F}$ such that $f(A) = c \cdot \mathrm{tr}(A)$, $\forall A \in \mathrm{End}(V)$.

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