PROBLEM SET 11, DUE APRIL 21

11. Inner product spaces.

Problem 11.1. *Define an inner product in the vector space* $\mathrm{Mat}(n \times n, \mathbb{C})$ *by*

$$\langle A, B\rangle := \mathrm{tr}((\overline{B})^t A).$$

Check that it is an inner product and find some orthonormal basis of $\mathrm{Mat}(n \times n, \mathbb{C})$.

Problem 11.2. *Let A be a linear operator in a complex finite-dimensional inner product space. Does there have to exist an orthonormal basis in which the matrix of A is upper-triangular?*

Problem 11.3. *Suppose that U and W are subspaces of a finite-dimensional inner product space V. Is it always true that*

$$(U + W)^\perp = U^\perp \cap W^\perp \quad ?$$

Problem 11.4. *Let U be a subspace of a finite-dimensional inner product space V. Show that there exists a unique projection operator $P_U \in \mathrm{End}(V)$ such that $\mathrm{im}(P_U) = U$ and $\ker(P_U) = (U)^\perp$. It is called the orthogonal projector onto U.*

Now suppose that $A \in \mathrm{End}(V)$ and the subspace U is A-invariant. Formulate an equivalent condition in terms of the projector P_U.

Problem 11.5. *Find a polynomial $p \in \mathbb{R}_2[x]$ such that for every $q \in \mathbb{R}_2[x]$ the following equality holds:*

$$\int_{-1}^{1} p(x)q(x)dx = q(0) \quad ?$$

Problem 11.6. *Consider the map $A : V \to V^*$ defined in Proposition 4.2 of Lecture 11. For a covector $f \in V^*$, the preimage $A^{-1}f$ is called its Riesz vector and is denoted by R_f.*

Suppose that U is a subspace of a finite-dimensional inner product space V, $f \in U^*$, $g \in V^*$ and $\forall u \in U$ we have that $f(u) = g(u)$. What can we say about the relation between the corresponding Riesz vectors R_f and R_g?

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