PROBLEM SET 13, DUE MAY 5

13. Various operators in inner product spaces.

Problem 13.1. *Does there exist a complex diagonalizable matrix such that it is not normal?*

Problem 13.2. *Let V be a finite-dimensional complex or real inner product space. Which of the following sets are subspaces of* $\mathrm{End}(V)$*:*

(1) *the set of normal operators;*
(2) *the set of unitary operators;*
(3) *the set of orthogonal operators;*
(4) *the set of Hermitian operators;*
(5) *the set of symmetric operators;*
(6) *the set of skew-Hermitian operators;*
(7) *the set of skew-symmetric operators;*
(8) *the set of non-negative operators?*

Problem 13.3. *Let V be a finite-dimensional complex or real inner product space. Which of the following sets are subgroups of* $\mathrm{GL}(V)$*:*

(1) *the set of normal operators;*
(2) *the set of unitary operators;*
(3) *the set of orthogonal operators;*
(4) *the set of Hermitian operators;*
(5) *the set of symmetric operators;*
(6) *the set of skew-Hermitian operators;*
(7) *the set of skew-symmetric operators;*
(8) *the set of positive definite operators?*

Problem 13.4. *Let V be a finite-dimensional inner product space. For an arbitrary non-zero vector $v \in V$ define the Householder reflection H_v by*

$$H_v(x) := x - \frac{2\langle x, v\rangle}{\langle v, v\rangle} v. \tag{13.1}$$

Show that every unitary (or orthogonal) operator can be written as a product of several Householder reflections.

Problem 13.5. *Suppose that V is a finite-dimensional complex inner product space, $A \in \mathrm{End}(V)$ and $\langle Ax, x\rangle$ is real for every $x \in V$. Does A have to be Hermitian?*

Problem 13.6. *Show that for every positive definite operator $A \in \mathrm{End}(v)$ there exists a unique positive definite operator $B \in \mathrm{End}(v)$ such that $B^2 = A$.*

Date: May 3, 2024.