PROBLEM SET 14, DUE MAY 12

14. Polar decomposition.

Problem 14.1. *Is it always true that the sum of two positive operators is positive? Is it always true that the product of two positive operators is positive?*

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Problem 14.2. *Suppose that A is a positive-definite operator in a finite-dimensional inner product space V, $\dim V > 1$. Is it true that the set*

$$\{X \in \mathrm{End}(V) \mid X^2 = A\}$$

can be finite as well as infinite depending on A? Can this set be empty?

Problem 14.3. *Can we say something more specific about the polar decomposition $A = UP$ if A is self-adjoint? Skew-Hermitian?*

Problem 14.4. *Can there be infinite number of polar decompositions $A = UP$ for some linear operator A? Can this happen if A is unitary?*

Problem 14.5. *Show that for an arbitrary matrix $A \in \mathrm{Mat}(n \times n, \mathbb{C})$ there exist unitary matrices $U_1, U_2 \in \mathrm{Mat}(n \times n, \mathbb{C})$ and a diagonal matrix $D \in \mathrm{Mat}(n \times n, \mathbb{C})$ with non-negative real values on the main diagonal such that*

$$A = U_1 D U_2.$$

Problem 14.6. *For every function $f : \mathbb{R} \to \mathbb{C}$ define a natural map $\widetilde{f}$ from the set of self-adjoint operators to the set of normal operators. Show that for every unitary operator U there exists a self-adjoint S such that*

$$\widetilde{\exp(iS)} = U.$$