LECTURE 1. VECTOR SPACES.

1. Preliminary Algebraic Definitions.

Definition 1.1. Consider a non-empty set G with a map mult : $G \times G \to G$. We are going to call such maps binary operations. In this case, we will denote the image mult(a, b) by ab. The pair $(G, mult)$ is called a monoid if the following conditions hold:

(1) $a(bc) = (ab)c, \quad \forall a, b, c \in G$;
(2) $\exists e \in G$ such that $ae = ea = a, \quad \forall a \in G$ (e is called the identity element).

Exercise 1.2. Show that the identity element e in a monoid is unique.

Definition 1.3. A monoid G is called a group if $\forall a \in G \, \exists b \in G$ such that

$$ab = ba = e. \tag{1.1}$$

An element b is called the inverse element of a and is denoted by a^{-1}.

Exercise 1.4. Show that for every group element, its inverse element is unique.

Definition 1.5. A group G such that

$$ab = ba, \ \forall a, b, \tag{1.2}$$

is called abelian. In an abelian group, the group operation is usually denoted by $+$, and the inverse element to an element a is denoted by $-a$.

Exercise 1.6. Does there exist a non-abelian group?

Definition 1.7. A set $\mathbb{F}$ with binary operations of multiplication and addition is called a field if the following conditions hold:

(1) $(\mathbb{F}, +)$ is an abelian group. We denote its identity element by 0 or by $0_{\mathbb{F}}$.
(2) $(\mathbb{F} \setminus \{0\}, \cdot)$ is an abelian group. We denote its identity element by 1 or by $1_{\mathbb{F}}$. Every group is a non-empty set, hence, $1 \neq 0$
(3) $a(b + c) = ab + ac$, $(a + b)c = ac + bc, \quad \forall a, b, c \in \mathbb{F}$.

Exercise 1.8. For a field $\mathbb{F}$ show that $a \cdot 0 = 0$, $\forall a \in \mathbb{F}$.

Exercise 1.9. Which of the following sets are fields with respect to standard operations:

$$\emptyset, \quad \{0\}, \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{R}_+, \quad \mathbb{R}, \quad \mathbb{C} \quad ?$$

Definition 1.10. A non-empty subset E of a field $\mathbb{F}$ is called a subfield if E is a field with respect to addition and multiplication defined in $\mathbb{F}$.

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2. Definition of a Vector Space.

Definition 2.1. Assume that $\mathbb{F}$ is a field, the elements of which we will call scalars. A non-empty set V with an operation of addition $+ : V \times V \to V, (x, y) \mapsto x + y$ and an operation of scalar multiplication scal : $\mathbb{F} \times V \to V$ $((\lambda, x) \mapsto \lambda x)$ is called a vector space over $\mathbb{F}$ if the following conditions hold:

(1) $(V, +)$ is an abelian group. We denote its identity element by 0 or by 0_V.
(2) $\lambda(x + y) = \lambda x + \lambda y, \quad \forall \lambda \in \mathbb{F}, \forall x, y \in V.$
(3) $(\lambda + \mu)x = \lambda x + \mu x, \quad \forall \lambda, \mu \in \mathbb{F}, \forall x \in V.$
(4) $(\lambda\mu)x = \lambda(\mu x), \quad \forall \lambda, \mu \in \mathbb{F}, \forall x \in V.$
(5) $1_{\mathbb{F}}\, x = x, \quad \forall x \in V.$

The notation here may be a bit confusing. For example, in condition (2), the symbol "+" in the LHS stands for the addition in the field $\mathbb{F}$, and in the RHS, the same symbol stands for the addition in V.

Exercise 2.2. For a vector space V over a field $\mathbb{F}$ show that $0_{\mathbb{F}}\, x = 0_V$, $\forall x \in V$.

Exercise 2.3. For a vector space V over a field $\mathbb{F}$ show that $(-1_{\mathbb{F}})\, x = -x$, $\forall x \in V$.

Exercise 2.4. Let V be a vector space over a field $\mathbb{F}$. Show that for $\forall \lambda \in \mathbb{F} \setminus \{0\}, \forall v, u \in V$ the equation

$$\lambda x + v = u \tag{2.1}$$

has a unique solution in V.

Definition 2.5. For a field $\mathbb{F}$ denote by

$$\mathbb{F}^n := \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mid a_j \in \mathbb{F}, \forall 1 \le j \le n \right\} \tag{2.2}$$

the set of ordered n-tuples of elements of $\mathbb{F}$. We define the component-wise addition on $\mathbb{F}^n$ and the multiplication of elements of $\mathbb{F}^n$ by the scalars of $\mathbb{F}$:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} := \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}, \quad \forall a_j, b_j \in \mathbb{F}; \tag{2.3}$$

$$\lambda \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} := \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{pmatrix}, \quad \forall a_j, \lambda \in \mathbb{F}. \tag{2.4}$$

Exercise 2.6. Which of the following pairs are vector spaces over corresponding fields:

(1) $\emptyset$ over $\mathbb{R}$;
(2) $\{0\}$ over $\mathbb{R}$
(3) $\mathbb{Z}$ over $\mathbb{Q}$;
(4) $\mathbb{Q}$ over $\mathbb{Z}$;
(5) $\mathbb{Q}$ over $\mathbb{R}$;
(6) $\mathbb{R}$ over $\mathbb{Q}$;
(7) $\mathbb{C}$ over $\mathbb{R}$;
(8) $\mathbb{C}^n$ over $\mathbb{C}$;
(9) $\mathbb{C}^n$ over $\mathbb{R}$?

Proposition 2.7. *A field is a vector space over its subfield.*

3. Subspaces, Direct Sums.

Definition 3.1. Assume that V is a vector space over a field $\mathbb{F}$. A non-empty subset U of V is called its subspace if:

(1) for every element $x, y \in U$ their sum $x + y \in U$;
(2) for every scalar $\lambda \in \mathbb{F}$ and for every element $x \in U$ the product $\lambda x \in U$.

Definition 3.2. Assume that V is a vector space over a field $\mathbb{F}$. A non-empty subset U of V is called its subspace if U is a vector space over $\mathbb{F}$ with respect to addition and multiplication by scalars defined in V.

Exercise 3.3. Show that the previous two definitions are equivalent.

It is widely agreed that if V is a vector space, then the notation $U \subset V$ means that U is a subspace of V unless specified otherwise.

Exercise 3.4. Is it always true that the intersection of an arbitrary family of subspaces of a vector space V is also a subspace?

Definition 3.5. Assume that $U_1, \dots, U_n$ are subspaces of the vector space V. Define a set

$$\sum_{j=1}^{n} U_j := \{\sum_{j=1}^{n} x_j \mid x_j \in U_j, \forall 1 \le j \le n\}. \tag{3.1}$$

It is called the sum or the join of the subspaces $U_1, \dots, U_n$. Another notation for it is $U_1 + U_2 + \dots + U_n$.

Example 3.6. *Suppose that*

$$V = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \mid a_1, a_2, a_3 \in \mathbb{R},\ a_1 + a_2 + a_3 = 0 \right\}, \tag{3.2}$$

$$U_1 = \left\{ \begin{pmatrix} b \\ -b \\ 0 \end{pmatrix} \mid b \in \mathbb{R} \right\}, \quad U_2 = \left\{ \begin{pmatrix} c \\ 0 \\ -c \end{pmatrix} \mid c \in \mathbb{R} \right\}, \quad U_3 = \left\{ \begin{pmatrix} 0 \\ d \\ -d \end{pmatrix} \mid d \in \mathbb{R} \right\}. \tag{3.3}$$

Then

$$V = U_1 + U_2 + U_3. \tag{3.4}$$

Exercise 3.7. Is it always true that $\sum_{j=1}^{n} U_j$ is also a subspace of the vector space V?

Definition 3.8. Assume that $U_1, \dots, U_n$ are subspaces of the vector space V. We are going to write

$$V = \bigoplus_{j=1}^{n} U_j \tag{3.5}$$

and say that V is a direct sum of $U_1, \dots, U_n$ if and only if $\forall x \in V\ \exists!(x_1, \dots, x_n)$ such that

(1) $x_j \in U_j$, $\forall 1 \le j \le n$;
(2) $x = \sum_{j=1}^{n} x_j$.

Exercise 3.9. Show that $V = U_1 \oplus U_2$ if and only if

(1) $V = U_1 + U_2$;
(2) $U_1 \cap U_2 = \{0\}$.

4. Quotient Spaces.

Definition 4.1. Let U be a subspace of a vector space V. Then we say that $x, y \in V$ are equivalent if and only if $x - y \in U$. We write

$$(x \sim_U y) \iff (x - y \in U). \tag{4.1}$$

Exercise 4.2. Show that $\sim_U$ is indeed an equivalence relation for every subspace U.

Exercise 4.3. Let U be a subspace of a vector space V over $\mathbb{F}$. Consider equivalence classes in V with respect to the relation $\sim_U$. If a class contains an element x we may denote that class by $x + U$. We define the operations of addition and multiplication by scalars on these equivalence classes by

$$(x + U) + (y + U) := (x + y) + U, \quad \lambda(x + U) := (\lambda x) + U. \tag{4.2}$$

Show that the operations are well-defined. In other words, the results do not depend on the choices of representatives. Also, show that these operations turn the set of equivalence classes into a vector space over $\mathbb{F}$.

Definition 4.4. The vector space of equivalence classes, constructed in the previous exercise, is called a quotient space and is denoted by V/U.

Chapters from the textbooks relevant to the lecture:

- Friedberg, Insel, Spence, Sections 1.2, 1.3;
- Roman, pp.17-19, 29-30, 35-38, 41-45, 87-89;
- Axler, ch. 1.