LECTURE 2. COMPLEX NUMBERS

Please consider each proposition in this lecture as an exercise.

1. Definitions

Definition 1.1. Define binary operations of addition and multiplication on the set $\mathbb{R} \times \mathbb{R}$ via the following formulas:

$$(a, b) + (c, d) = (a + c, b + d); \tag{1.1}$$

$$(a, b) \cdot (c, d) = (ac - bd, ad + bc). \tag{1.2}$$

We denote the set $\mathbb{R} \times \mathbb{R}$ with these two operations by $\mathbb{C}$ and call its elements complex numbers.

There is an alternative, more common notation for complex numbers. We denote a pair (a, b) for $a, b \in \mathbb{R}$ by $a + bi$. Then the equalities (1.1) and (1.2) take the form

$$(a + bi) + (c + di) = (a + c) + (b + d)i; \tag{1.3}$$

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i. \tag{1.4}$$

We may interpret these operations as adding an element i to the field $\mathbb{R}$ with the property $i \cdot i = -1$, and then obtaining all elements needed for closure under addition and multiplication, provided that the associativity, commutativity, and distributivity laws hold. For this reason, we denote $(0, 0) = 0 + 0i$ by 0 and $(1, 0) = 1 + 0i$ by 1.

We may consider $\mathbb{R}$ as a subset of $\mathbb{C}$ consisting of elements $a + 0i$ for $a \in \mathbb{R}$.

Exercise 1.2. Compute:
a) $-i(2 - i)$;
b) $(1 + i)(1 - i)$;
c) $(\sqrt{3} + i)^3$.

Definition 1.3. We consider several important maps on $\mathbb{C}$. Define the real part $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$ and the imaginary part $\mathrm{Im} : \mathbb{C} \to \mathbb{R}$ by

$$\mathrm{Re(a + bi)} = a; \tag{1.5}$$

$$\mathrm{Im(a + bi)} = b. \tag{1.6}$$

Also, define the absolute value $|\cdot| : \mathbb{C} \to \mathbb{R}_{\geq 0}$ by

$$|a + bi| = \sqrt{a^2 + b^2}. \tag{1.7}$$

We also consider the complex conjugation map from $\mathbb{C}$ to itself ($z \mapsto \overline{z}$), defined by

$$\overline{a + bi} = a - bi. \tag{1.8}$$

Proposition 1.4.

$$|z|^2 = (\mathrm{Re(z)})^2 + (\mathrm{Im(z)})^2 = \mathrm{z} \cdot \overline{\mathrm{z}}, \quad \forall \mathrm{z} \in \mathbb{C}. \tag{1.9}$$

Proposition 1.5.

$$\overline{z + w} = \overline{z} + \overline{w}; \tag{1.10}$$

$$\overline{zw} = \overline{z} \cdot \overline{w}, \quad \forall z, w \in \mathbb{C}. \tag{1.11}$$

Proposition 1.6.

$$|zw| = |z| \cdot |w|, \quad \forall z, w \in \mathbb{C}. \tag{1.12}$$

Date: February 17, 2025.

Exercise 1.7. Show that for all $z \in \mathbb{C} \setminus \{0\}$, there exists $w \in \mathbb{C}$ such that

$$zw = wz = 1. \tag{1.13}$$

Proposition 1.8. $\mathbb{C}$ *is a field.*

2. Exponential Function

Lemma 2.1. *Suppose* $a, b \in \mathbb{R}$ *and* $a^2 + b^2 = 1$. *Then there exists a unique* $\varphi \in (-\pi, \pi]$ *such that*

$$a = \cos(\varphi), \quad b = \sin(\varphi). \tag{2.1}$$

Definition 2.2. From the previous result, we get that for all $z \in \mathbb{C} \setminus \{0\}$, there exists a unique $\varphi \in (-\pi, \pi]$ such that

$$z = |z|(\cos(\varphi) + i\sin(\varphi)). \tag{2.2}$$

We denote

$$\mathrm{Arg(z)} = \varphi. \tag{2.3}$$

Arg(z) is called the argument of the complex number z.

Exercise 2.3. Show that

$$(\cos(\varphi) + i\sin(\varphi))(\cos(\psi) + i\sin(\psi)) = \cos(\varphi + \psi) + i\sin(\varphi + \psi), \quad \forall \varphi, \psi \in \mathbb{R}. \tag{2.4}$$

Definition 2.4. Define a map $\exp : \mathbb{C} \to \mathbb{C} \setminus \{0\}$ by the formula

$$\exp(a + bi) := \exp(a)(\cos(b) + i\sin(b)). \tag{2.5}$$

Proposition 2.5.

$$\exp(z + w) = \exp(z) \cdot \exp(w), \forall z, w \in \mathbb{C}. \tag{2.6}$$

Exercise 2.6. Let $n \in \mathbb{Z}_+$. Solve the equation

$$z^n = 1 \tag{2.7}$$

over $\mathbb{C}$.

Exercise 2.7. Let $n \in \mathbb{Z}_+$. Solve the equation

$$z^n = -2 \tag{2.8}$$

over $\mathbb{C}$.