LECTURE 3. LINEAR INDEPENDENCE

Please consider each proposition in this lecture as an exercise.
Throughout this lecture, we will assume that $\mathbb{F}$ is a field and V is a vector space over $\mathbb{F}$.

1. Linear Span

Exercise 1.1. Does there exist a subspace U of the vector space $\mathbb{R}^3$ such that it contains the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \tag{1.1}$$

and U is a subspace of any other subspace with this property? In other words, we are looking for the smallest subspace containing

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}. \tag{1.2}$$

The result of this exercise inspires the following definition.

Definition 1.2. For a list of vectors $(v_1, \dots, v_n)$ we define its linear span as the set

$$\boxed{\langle v_1, \dots, v_n \rangle := \{\sum_{j=1}^{n} a_j v_j \mid a_1, \dots, a_n \in \mathbb{F}\}.} \tag{1.3}$$

The expression $\sum_{j=1}^{n} a_j v_j$ is called a linear combination of the vectors $v_1, \dots, v_n$. We consider a list of vectors instead of a set to allow a vector to appear in that list more than once.

We allow a list to be empty. In that case,

$$\langle \emptyset \rangle := \{0\}. \tag{1.4}$$

Exercise 1.3. Describe

$$\left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\rangle \tag{1.5}$$

in the vector space $\mathbb{R}^3$.

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Proposition 1.4. *Show that $\langle v_1, \ldots, v_n \rangle$ is a subspace of V.*

Proposition 1.5. *Suppose that $(v_1, \ldots, v_n)$ is a list of vectors from a vector space V, W is a subspace of V such that $v_i \in W$, $\forall 1 \leq i \leq n$. Then*

$$\langle v_1, \ldots, v_n \rangle \subset W. \tag{1.6}$$

2. Linear Independence

Exercise 2.1. Do there exist vectors $v_1, v_2, v_3 \in \mathbb{R}^3$ such that

$$\langle v_1, v_2, v_3 \rangle = \langle v_1, v_2 \rangle? \tag{2.1}$$

The previous exercise partially motivates the following definition.

Definition 2.2. A list of vectors $(v_1, \ldots, v_n)$ is called linearly dependent if $\exists (a_1, \ldots, a_n) \in \mathbb{F}^n$ such that:

(1) $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$;
(2)

$$\sum_{j=1}^{n} a_j v_j = 0. \tag{2.2}$$

If a list of vectors $(v_1, \ldots, v_n)$ is not linearly dependent, it is called linearly independent.

Exercise 2.3. Show that a linearly independent list of vectors must be a set.

Exercise 2.4. Can a linearly independent list of vectors contain the zero vector?

Exercise 2.5. Is it true that a subset of a linearly independent set of vectors is also linearly independent?

Proposition 2.6. *Suppose that* $\{v_1, v_2, \ldots, v_n\}$ *is a linearly independent subset of a vector space* V *and* $u \in V$*. Then the list of vectors* $(v_1, v_2, \ldots, v_n, u)$ *is linearly dependent iff*

$$u \in \langle v_1, \ldots, v_n \rangle. \tag{2.3}$$

Proposition 2.7 (Linear Dependence Lemma)**.** *Suppose that a list of vectors* $(u_1, \ldots, u_m)$ *is linearly dependent, and* $u_1 \neq 0$*. Show that* $\exists 2 \leq j \leq m$ *such that the following hold:*

(1) $u_j \in \langle u_1, \ldots, u_{j-1} \rangle$*;*
(2) *if the* j*-th term is removed from* $(u_1, \ldots, u_m)$*, then the span of the remaining list equals* $\langle u_1, \ldots, u_m \rangle$*.*

Now, we can prove a crucial result that will allow us to introduce the notion of the dimension of a vector space later.

Proposition 2.8 (Replacement Theorem)**.** *Assume that*

$$\langle v_1, \ldots, v_n \rangle = V, \tag{2.4}$$

and a subset $\{u_1, \ldots, u_m\}$ *of* V *is linearly independent. Then* $m \leq n$*.*

Suggested reading:

- Friedberg, Insel, Spence, Sections 1.4 - 1.6;
- Roman, pp.44 - 48;
- Axler, ch. 2.