LECTURE 4. BASIS OF A VECTOR SPACE

Please consider each proposition in this lecture as an exercise.
Throughout this lecture, we will assume that $\mathbb{F}$ is a field and V is a vector space over $\mathbb{F}$.

1. Equivalent Definitions

Definition 1.1. A set of vectors $\{v_1, \dots, v_n\}$ is called a basis of a vector space V if the following two conditions hold:

(1) The set $\{v_1, \dots, v_n\}$ is linearly independent;
(2) $\langle v_1, \dots, v_n \rangle = V$.

Exercise 1.2. Consider the vector space $\mathbb{R}^3$. Show that the set of vectors

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{1.1}$$

is a basis of $\mathbb{R}^3$. Is the set

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{1.2}$$

also a basis of $\mathbb{R}^3$?

Let us consider other equivalent ways to look at the notion of a basis.

Proposition 1.3. *Let $\{v_1, \dots, v_n\}$ be a set of vectors in V. The following conditions are equivalent:*

(1) *The set $\{v_1, \dots, v_n\}$ is a basis of V.*

(2) *$\{v_1, \dots, v_n\}$ is a maximal linearly independent set, that is, $\{v_1, \dots, v_n\}$ is linearly independent, but if any vector is added, the set becomes linearly dependent.*

(3) *$\{v_1, \dots, v_n\}$ is a minimal spanning set of V, that is, $\{v_1, \dots, v_n\}$ spans V but any proper subset of $\{v_1, \dots, v_n\}$ does not span V.*

(4)
$$V = \bigoplus_{j=1}^{n} \langle v_j \rangle. \tag{1.3}$$

(5) $\forall x \in V \; \exists! (a_1, \dots, a_n) \in \mathbb{F}^n$ *such that*
$$x = \sum_{j=1}^{n} a_j v_j. \tag{1.4}$$

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Let us list the corollaries of the Replacement Theorem for the case when there exists a finite basis.

Proposition 1.4. *Let $\{v_1, \ldots, v_n\}$ be a basis of V. Then*

(1) *Every list of vectors from V that has the length greater than n is linearly dependent;*

(2) *If a set $\{t_1, \ldots, t_m\}$ is linearly independent, then $m \leq n$;*

(3) *If $\langle u_1, \ldots, u_m \rangle = V$ then $m \geq n$;*

(4) *If $\langle w_1, \ldots, w_n \rangle = V$ then the set $\{w_1, \ldots, w_n\}$ is a basis of V;*

(5) *If a set $\{x_1, \ldots, x_n\}$ is linearly independent, then it is a basis of the vector space V;*

(6) *If a set $\{y_1, \ldots, y_m\}$ is a basis of V then $m = n$.*

2. Dimension of a Vector Space and Its Properties

Definition 2.1. Let $\{v_1, \ldots, v_n\}$ be a basis of V. Then we say that V is a finite-dimensional space, denoted by $\dim V < \infty$. The number n is called the dimension of the vector space V. In this case, we will write $\dim V = n$.

Note that $\dim\{0\} = 0$.

Proposition 2.2. *Let U be a proper subspace of a finite-dimensional vector space V. Then U is also finite-dimensional and $\dim U < \dim V$.*

Next, we are going to discuss how we can construct a basis from existing subsets of vectors (Propositions 2.3 and 2.5).

Proposition 2.3. *Assume that $\dim V = n$ and $\{u_1, \ldots, u_m\}$ is a linearly independent subset of V. Show that there exist vectors $w_1, \ldots, w_{n-m}$ such that $\{u_1, \ldots, u_m, w_1, \ldots, w_{n-m}\}$ is a basis of V.*

Proposition 2.4. *Let U be a subspace of a finite-dimensional vector space V. Then there exists a subspace W such that*

$$V = U \oplus W. \tag{2.1}$$

Proposition 2.5. *Assume that $\dim V = n$ and*

$$\langle u_1, \ldots, u_m \rangle = V. \tag{2.2}$$

Then there exists a subset of $\{u_1, \ldots, u_m\}$ such that it is a basis of V.

3. Dimension of the Intersection of two subspaces

Proposition 3.1. *Let U and W be subspaces of a finite-dimensional vector space V. Then*

$$\boxed{\dim(W+U) = \dim W + \dim U - \dim(W \cap U).} \tag{3.1}$$

Proposition 3.2. *Suppose that* $\dim V < \infty$ *and*

$$V = \bigoplus_{j=1}^{k} U_j. \tag{3.2}$$

Then

$$\dim V = \sum_{j=1}^{k} \dim U_j. \tag{3.3}$$

Chapters from the textbooks relevant to the lecture:

- Friedberg, Insel, Spence, Section 1.6;
- Roman, pp.48 – 51;
- Axler, ch. 2.