LECTURE 5. LINEAR MAPS

Throughout this lecture, we will assume that $\mathbb{F}$ is a field, and all vector spaces are defined over that same field. Please consider the proof of every proposition in this lecture to be an exercise.

1. Types of Linear Maps

Definition 1.1. Assume that V and U are vector spaces over the same field $\mathbb{F}$. A map $A : V \to U$ is called a linear map (or, a linear transformation, or a homomorphism) if

$$\boxed{A(x+y) = Ax + Ay, \quad A(\lambda x) = \lambda Ax, \quad \forall x, y \in V, \forall \lambda \in \mathbb{F}.} \tag{1.1}$$

The set of all linear maps from V to U is denoted by $\mathrm{Hom}(V, U)$.

Exercise 1.2. Let $A \in \mathrm{Hom}(V, U)$. Is it always true that

$$A(0) = 0? \tag{1.2}$$

Exercise 1.3. Suggest an example of vector spaces V and U over the same field and a map $A : V \to U$ such that

$$A(x+y) = Ax + Ay, \quad \forall x, y \in V, \tag{1.3}$$

but A is not a linear map.

Exercise 1.4. Suggest an example of vector spaces V and U over the same field and a map $A : V \to U$ such that

$$A(\lambda x) = \lambda Ax, \quad \forall x \in V, \forall \lambda \in \mathbb{F}, \tag{1.4}$$

but A is not a linear map.

Definition 1.5. We define pointwise addition and scalar multiplication in $\mathrm{Hom}(V, U)$:

$$(A+B)(x) := Ax + Bx, \quad (\lambda A)(x) := \lambda(Ax), \quad \forall A, B \in \mathrm{Hom}(V, U), \forall x \in V, \forall \lambda \in \mathbb{F}. \tag{1.5}$$

Exercise 1.6. Show that if $A, B \in \mathrm{Hom}(V, U)$, then $A + B, \lambda A \in \mathrm{Hom}(V, U)$.

Exercise 1.7. Show that $\mathrm{Hom}(V, U)$ is a vector space with respect to those operations.

Definition 1.8. We are going to use the following terms and notation:

(1) A linear map from a vector space to itself is called an endomorphism or a linear operator. The set of all linear operators on V is denoted by $\mathrm{End}(V)$. A map $\mathrm{Id} \in \mathrm{End}(V)$ defined by
$$\mathrm{Id}(x) := x, \quad \forall x \in V \tag{1.6}$$
is called the identity map.
(2) A linear map is called a monomorphism or an embedding if it is injective.
(3) A linear map is called an epimorphism if it is surjective.
(4) A linear map is called an isomorphism if it is bijective. The set of all isomorphisms from a vector space V to a vector space U is denoted by $\mathrm{Iso}(V, U)$. If there exists an isomorphism from V to U, we will write $V \cong U$.
(5) A linear operator is called an automorphism if it is bijective. The set of all automorphisms of a vector space V is denoted by $\mathrm{GL}(V)$ (which stands for the "General Linear (Group)") or by $\mathrm{Aut}(V)$.

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Exercise 1.9. Which of the sets $\mathrm{End}(V)$, $\mathrm{Iso}(V,U)$, and $\mathrm{GL}(V)$ are vector spaces with respect to the pointwise addition and the pointwise multiplication by scalars? Is the set of monomorphisms a subspace of $\mathrm{Hom}(V,U)$? Is the set of epimorphisms a subspace of $\mathrm{Hom}(V,U)$?

Definition 1.10. Suppose that V, U, W are finite-dimensional spaces, $A \in \mathrm{Hom}(V,U), B \in \mathrm{Hom}(U,W)$. We may also consider an operation of composition of linear maps B and A. We are going to denote this composition of linear maps by BA:

$$BA(x) := B(A(x)), \quad \forall x \in V. \tag{1.7}$$

We use this notation because the operation of the composition of linear operators shares several nice properties with multiplication.

Exercise 1.11. Is $\mathrm{End}(V)$ a group with respect to this operation? Is $\mathrm{GL}(V)$ a group with respect to the operation of composition? Is it always true that

$$A(B+C) = AB + AC, \quad \forall A, B.C \in \mathrm{End}(V)? \tag{1.8}$$

2. Image and Kernel of a Linear Map

Definition 2.1. For an arbitrary $A \in \mathrm{Hom}(V,U)$ we define

$$\ker(A) := \{x \in V \mid Ax = 0\}, \quad \mathrm{im}(A) := \{y \in U \mid \exists x \in V \text{ such that } Ax = y\}. \tag{2.1}$$

The set $\ker(A)$ is called the kernel (or the null space) of A. The set $\mathrm{im}(A)$ is called the image (or the range) of A.

Exercise 2.2. Find the kernel and the image of the identity map.

Proposition 2.3. *For an arbitrary $A \in \mathrm{Hom}(V,U)$ the set $\ker(A)$ is a subspace of the vector space V, $\mathrm{im}(A)$ is a subspace of the vector space U.*

Exercise 2.4. Show that

$$(A \text{ is a monomorphism}) \quad \Longleftrightarrow \quad (\ker A = \{0\}). \tag{2.2}$$

Definition 2.5. Assume that V and U are finite-dimensional vector spaces. For a linear map $A \in \mathrm{Hom}(V,U)$ we put

$$\mathrm{rk}(A) := \dim(\mathrm{im}(A)). \tag{2.3}$$

The number $\mathrm{rk}(A)$ is called the rank of the homomorphism A, the number $\dim(\ker(A))$ is called its nullity.

Exercise 2.6. Describe all linear maps $A \in \mathrm{Hom}(V,U)$ such that

$$\mathrm{rk}(A) = 0. \tag{2.4}$$

3. Isomorphisms

The following two propositions will provide another way to look at isomorphisms.

Proposition 3.1. *A linear map $A \in \mathrm{Hom}(V,U)$ is an isomorphism if and only if $\ker(A) = \{0\}$ and $\mathrm{im}(A) = U$. Then the inverse map A^{-1} is also linear and, therefore, $A^{-1} \in \mathrm{Iso}(U,V)$.*

Proposition 3.2. *Suppose that V and U are vector spaces and $A \in \mathrm{Hom}(V,U)$. Then*

$$(A \in \mathrm{Iso}(V,U)) \quad \Longleftrightarrow \quad (\exists B \in \mathrm{Hom}(U,V) \text{ such that } AB = BA = \mathrm{Id}). \tag{3.1}$$

If we can pick a finite basis in a vector space, then we may construct an isomorphism of that space with one of the standard vector spaces $\mathbb{F}^n$.

Proposition 3.3. *Suppose that V is a finite-dimensional space, and $\alpha = (v_1, \dots, v_n)$ is a basis of V. We assume that the basis is ordered; i.e., the order of the elements in the set matters. We use the parentheses instead of the curly brackets to stress that we consider an ordered sequence.*

As we know, $\forall x \in V \; \exists! (a_1, \dots, a_n) \in \mathbb{F}^n$ such that

$$x = \sum_{j=1}^{n} a_j v_j. \tag{3.2}$$

The numbers $(a_1, \dots, a_n)$ are called the coordinates of the vector x in the basis α. Let us define a map $\varphi_\alpha : V \to \mathbb{F}^n$ by setting

$$\varphi_\alpha(x) := \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}. \tag{3.3}$$

Then the map $\varphi_\alpha \in \mathrm{Iso}(V, \mathbb{F}^n)$.

This claim allows us to provide the following classification of finite-dimensional vector spaces over the same field up to isomorphism.

Proposition 3.4. *Assume that V and U are finite-dimensional vector spaces. There exists an isomorphism between V and U (or, we may say that V and U are isomorphic) if and only if $\dim V = \dim U$.*

Thus, to find out whether two finite-dimensional vector spaces are isomorphic, it is sufficient to check only their dimensions. In other words, consider an equivalence relation of being isomorphic on finite-dimensional vector spaces over a fixed vector space $\mathbb{F}$. Then, every equivalence class consists of all finite-dimensional vector spaces of the same dimension, including $\mathbb{F}^n$ for some n.

Chapters from the textbooks relevant to the lecture:

- Friedberg, Insel, Spence, Sections 2.1, 2.4;
- Roman, pp.59 - 63;
- Axler, ch. 3.