LECTURE 6. MATRICES OF LINEAR MAPS

Throughout this lecture, we will assume that $\mathbb{F}$ is a field, and vector spaces are defined over that same field. Please consider the proof of every proposition in this lecture to be an exercise.

1. Definition of a Matrix of a Linear Map

Proposition 1.1. *Suppose that V and U are vector spaces over the field $\mathbb{F}$, and an ordered list $(v_1, \dots, v_n)$ is a basis of the vector space V. Then, for every list of vectors $(y_1, \dots, y_n)$ from U there exists a unique map $A \in \mathrm{Hom}(V, U)$ such that $Av_j = y_j$, $\forall 1 \le j \le n$.*

From Proposition 1.1, we can deduce the following useful corollary.

Proposition 1.2. *Suppose that $B, C \in \mathrm{Hom}(V, U)$, $(v_1, \dots, v_n)$ is a basis of the vector space V, and*

$$Bv_j = Cv_j, \forall 1 \le j \le n. \tag{1.1}$$

Then $B = C$.

The following definition establishes how we can represent linear maps using matrices.

Definition 1.3. Suppose that $\alpha = (v_1, \dots, v_n)$ is an ordered basis of a vector space V, $\beta = (u_1, \dots, u_m)$ is an ordered basis of a vector space U, and $A \in \mathrm{Hom}(V, U)$. Then there exists a list (a_i^j), $1 \le i \le n$, $1 \le j \le m$, of elements of the field $\mathbb{F}$ such that

$$\boxed{Av_i = \sum_{j=1}^{m} a_i^j u_j, \quad \forall 1 \le i \le n.} \tag{1.2}$$

The matrix filled with (a_i^j) in the following order

$$\begin{pmatrix} a_1^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^m & \dots & a_n^m \end{pmatrix} \tag{1.3}$$

is called the matrix of the map A (in the bases α and β), and is denoted by the same letter A or by $[A]_\alpha^\beta$. The set of $m \times n$ matrices with entries from the field $\mathbb{F}$ is denoted by $\mathrm{Mat}(m \times n, \mathbb{F})$.

Exercise 1.4. Suppose that we picked ordered bases in finite-dimensional vector spaces V and U, and $A, B \in \mathrm{Hom}(V, U)$. Express the entries of the matrices of the linear maps $A + B$ and λA using the entries of A and B.

Exercise 1.5. Show that the set $\mathrm{Mat}(m \times n, \mathbb{F})$ is a vector space with respect to entrywise addition and scalar multiplication.

Exercise 1.6. Suppose that V and U are finite-dimensional vector spaces. Show that the vector spaces $\mathrm{Hom}(V, U)$ and $\mathrm{Mat}(\dim U \times \dim V, \mathbb{F})$ are isomorphic.

Exercise 1.7. Assume that V and U are finite-dimensional vector spaces. Show that

$$\boxed{\dim(\mathrm{Hom}(V, U)) = \dim V \cdot \dim U.} \tag{1.4}$$

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2. The Matrix Multiplication

Now, we will establish the rule for multiplication of matrices that corresponds to the composition of linear maps.

Definition 2.1. Suppose that V, U, W are finite-dimensional spaces with ordered bases α, β, γ, respectively, and $A \in \mathrm{Hom}(V, U), B \in \mathrm{Hom}(U, W)$. We define the product of matrices via the formula

$$\boxed{[B]^{\gamma}_{\beta} \cdot [A]^{\beta}_{\alpha} := [BA]^{\gamma}_{\alpha}.} \tag{2.1}$$

Proposition 2.2 (The Row-Column rule)**.** *Suppose that V, U, W are finite-dimensional spaces with ordered bases α, β, γ respectively, $A \in \mathrm{Hom}(V, U), B \in \mathrm{Hom}(U, W)$, and*

$$[A]^{\beta}_{\alpha} = (a^q_p)_{1\le p\le k, 1\le q\le m}, \quad [B]^{\gamma}_{\beta} = (b^s_r)_{1\le r\le m, 1\le s\le n}, \quad [BA]^{\gamma}_{\alpha} = (c^j_i)_{1\le i\le k, 1\le j\le n}. \tag{2.2}$$

Show that

$$c^j_i = \sum_{t=1}^{m} b^j_t a^t_i, \quad \forall 1 \le i \le k, \forall 1 \le j \le n. \tag{2.3}$$

Exercise 2.3. Assume that X is an $m \times n$ matrix, Y is a $p \times m$ matrix. Show that Definition 2.1 determines a unique product $Y \cdot X$, which is a $p \times n$ matrix.

Using the Row-Column rule notation, we may rewrite the Definition 1.3 as follows

$$(Av_1, \dots, Av_n) = (u_1, \dots, u_m) \begin{pmatrix} a^1_1 & \dots & a^1_n \\ \vdots & \ddots & \vdots \\ a^m_1 & \dots & a^m_n \end{pmatrix}. \tag{2.4}$$

Exercise 2.4. Is the matrix multiplication associative?

Exercise 2.5. Does the distributivity law hold for the matrix multiplication and addition?

Exercise 2.6. Is the set $\mathrm{Mat}(m \times n, \mathbb{F})$ a group with respect to multiplication?

3. The First Isomorphism Theorem

Next, we will introduce a version of the fundamental theorem of homomorphisms for vector spaces.

Proposition 3.1 (The First Isomorphism Theorem)**.** *For an arbitrary $A \in \mathrm{Hom}(V, U)$, define the map p from the quotient space $V/\ker(A)$ to the vector space U by*

$$p(x + \ker(A)) := Ax, \quad \forall x \in V. \tag{3.1}$$

Then p is a well-defined linear map. Moreover, p is an isomorphism between the quotient space $V/\ker(A)$ and $\mathrm{im}(A)$.

Now, let us deduce a simple formula for the dimension of a quotient space.

Proposition 3.2. *Suppose that U is a subspace of a finite-dimensional space V, $\{u_1, \dots, u_m\}$ is a basis of U, and $\{u_1, \dots, u_m, w_1, \dots, w_k\}$ is a basis of V. Then $\{w_1 + U, \dots, w_k + U\}$ is a basis of V/U. In particular,*

$$\boxed{\dim(V/U) = \dim V - \dim U.} \tag{3.2}$$

Propositions 3.1 and 3.2 imply the following helpful formula.

Proposition 3.3 (The Rank Plus Nullity Theorem)**.** *Assume that V and U are finite-dimensional vector spaces. For a linear map $A \in \mathrm{Hom}(V, U)$, we get the identity*

$$\boxed{\mathrm{rk}(A) + \dim(\ker(A) = \dim V.} \tag{3.3}$$

Chapters from the textbooks relevant for this lecture:

- Friedberg, Insel, Spence, Sections 2.1 - 2.3;
- Axler, Sections 3B, 3C.