PROBLEM SET 1, DUE FEBRUARY 15

1. Vector Spaces

Definition 1.1. A polynomial in the variable x is a formal sum of the form

$$p = \sum_{j=0}^{k} a_j x^j, \quad k \geq 0,\ a_j \in \mathbb{F}. \tag{1.1}$$

We assume that two such sums are equal if they differ only by some monomials with zero coefficients. If $a_k \neq 0$ we say that the degree of the polynomial p denoted by $\deg(p) := k$. In this case, we will call a_k the leading coefficient of the polynomial p. If all coefficients of the polynomial are zero, then we define $\deg(0) := -\infty$. The set of polynomials is a vector space with respect to standard addition and scalar multiplication. We denote this vector space by $\mathbb{F}[x]$. Also, we put

$$\mathbb{F}_n[x] := \{p \in \mathbb{F}[x] \mid \deg(p) \leq n\}. \tag{1.2}$$

With every polynomial $p \in \mathbb{F}[x]$ we associate a function $p : \mathbb{F} \to \mathbb{F}$.

Note that $\forall a \in \mathbb{F}, a \neq 0$ we have that $\deg(a) = 0$.

Problem 1.2. *Which of the following sets are subspaces of the space of polynomials with real coefficients $\mathbb{R}[x]$:*

(1) $\{p \in \mathbb{R}[x] \mid p(1) = 0\}$;
(2) $\{p \in \mathbb{R}[x] \mid p(1) = 1\}$;
(3) $\{p \in \mathbb{R}[x] \mid p(0) = 0, p(1) = 0\}$;
(4) $\{p \in \mathbb{R}[x] \mid p(0)p(1) = 0\}$?

Problem 1.3. *Suppose that U_1 and U_2 are subspaces of a vector space V. Are the conditions*

$$U_1 + U_2 \subset U_1$$

and

$$U_2 \subset U_1$$

equivalent?

Problem 1.4. *Consider the operation of addition on the set of subspaces of a fixed vector space V. Is the set of subspaces a monoid under this operation?*

Problem 1.5. *Is it true that for any subspaces U_1, U_2, U_3 of V the following equality holds*

$$(U_1 + U_2) \cap U_3 = U_1 \cap U_3 + U_2 \cap U_3?$$

Problem 1.6. *Denote by L the vector space of all maps from $\mathbb{R}$ to itself. Consider its subspaces*

$L_{even} := \{f : \mathbb{R} \to \mathbb{R} \mid f(-x) = f(x), \forall x \in \mathbb{R}\}$ *and* $L_{odd} := \{f : \mathbb{R} \to \mathbb{R} \mid f(-x) = -f(x), \forall x \in \mathbb{R}\}$.

Is it true that $L = L_{even} \oplus L_{odd}$?

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Problem 1.7. *Consider*

$$U = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mid \sum_{j=1}^{n} j x_j = 0 \right\} \subset \mathbb{R}^n.$$

Does there exist a subspace W such that $\mathbb{R}^n = U \oplus W$?

Problem 1.8. *Let $\mathbb{R}[x]$ be the space of polynomials with real coefficients and consider its subspace*

$$U = \{p \in \mathbb{R}[x] \mid p(1) = 0\}.$$

Does there exist a subspace W such that $\mathbb{R}[x] = U \oplus W$?

Problem 1.9. *Do there exist a vector space V and its subspaces U_1, U_2 and U_3 such that*

(1) $V = U_1 \oplus U_2$;
(2) $V = U_1 \oplus U_3$;
(3) $U_2 \neq U_3$?

Problem 1.10. *Suppose that U_1, U_2 and U_3 are subspaces of a vector space V. Is it true that $V = U_1 \oplus U_2 \oplus U_3$ if and only if the following conditions hold:*

(1) $V = U_1 + U_2 + U_3$;
(2) $U_1 \cap U_2 = \{0\}$;
(3) $U_1 \cap U_3 = \{0\}$;
(4) $U_2 \cap U_3 = \{0\}$?

Problem 1.11. *Suppose that $\mathbb{R}[x]$ is the space of polynomials with real coefficients and*

$$U = \{p \in \mathbb{R}[x] \mid p(1) = 0\}.$$

Describe a natural bijection between the quotient space $\mathbb{R}[x]/U$ and the field $\mathbb{R}$.