# PROBLEM SET 3

## 3. Linear Independence

**Problem 3.1.** *Let $(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+m})$ be a list of vectors from a vector space $V$. Determine whether it is always true that*

$$\langle v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+m} \rangle = \langle v_1, \ldots, v_k \rangle + \langle v_{k+1}, \ldots, v_{k+m} \rangle. \tag{3.1}$$

**Problem 3.2.** *Let $(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+m})$ be a list of vectors from a vector space $V$. Is it always true that*

$$\langle v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+m} \rangle = \langle v_1, \ldots, v_k \rangle \oplus \langle v_{k+1}, \ldots, v_{k+m} \rangle? \tag{3.2}$$

**Problem 3.3.** *Let $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+m}\}$ be a set of vectors from a vector space $V$. Determine whether it is always true that*

$$\langle v_1, \ldots, v_k \rangle \cap \langle v_{k+1}, \ldots, v_{k+m} \rangle = \{0\}. \tag{3.3}$$

**Problem 3.4.** *Consider the vector space $\mathbb{R}^3$. Is it true that*

$$\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\rangle = \mathbb{R}^3? \tag{3.4}$$

**Problem 3.5.** *Consider the vector space $\mathbb{R}^3$. Is the set of vectors*

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{3.5}$$

*linearly independent?*

**Problem 3.6.** *Consider the vector space $\mathbb{R}^3$. Determine if the set of vectors*

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{3.6}$$

*linearly independent.*

**Problem 3.7.** *Consider the vector space $\mathbb{R}^3$. Is the set of vectors*

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \tag{3.7}$$

*linearly independent?*

**Problem 3.8.** *Consider the vector space $\mathbb{R}^3$. Determine if the set of vectors*

$$\left\{ \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\} \tag{3.8}$$

*linearly independent.*

---

*Date*: March 15, 2025.

**Problem 3.9.** *Consider the vector space $\mathbb{R}_2[x]$. Is the set of vectors*

$$\left\{x^2 - x, x^2 + 2x - 3, -x + 1\right\} \tag{3.9}$$

*linearly independent?*

**Problem 3.10.** *Let $u$ and $v$ be distinct vectors in a vector space $V$. Show that $\{u, v\}$ is linearly dependent if and only if $u$ or $v$ is a multiple of the other.*

**Problem 3.11.** *Provide an example of three linearly dependent vectors in $\mathbb{R}^3$ such that none of the three is a multiple of another.*

**Problem 3.12.** *Prove that a set of vectors $\{u, v, w\}$ is linearly independent if and only if the set $\{u + v, v + w, u + w\}$ is linearly independent.*

**Problem 3.13.** *Suppose that vectors $u, v, w, x \in V$ are such that*

$$x \notin \langle u, v, w\rangle. \tag{3.10}$$

*Then a set of vectors $\{u, v, w\}$ is linearly independent if and only if the set $\{u + x, v + x, w + x\}$ is linearly independent.*