PROBLEM SET 4

4. Basis of a Vector Space

Problem 4.1. *Is the set*

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\} \tag{4.1}$$

a basis of $\mathbb{R}^3$*?*

Problem 4.2. *Is the set*

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \tag{4.2}$$

a basis of $\mathbb{R}^4$*?*

Problem 4.3. *Consider the vector space* $\mathbb{R}_2[x]$*. Is the set of vectors*

$$\left\{x^2 - x + 1, x^2 + x - 3, x^2 - 2, x + 1\right\} \tag{4.3}$$

linearly independent?

Problem 4.4. *Is it true that*

$$\left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\rangle = \mathbb{R}^4? \tag{4.4}$$

Problem 4.5. *Suppose that* v, u, w *are distinct vectors of a vector space* V *and* $v, v+u, v+u+w$ *are distinct vectors from* V*. Show that the set* $\{v, u, w\}$ *is a basis of the vector space* V *if and only if the set* $\{v, v+u, v+u+w\}$ *is a basis of* V*.*

Problem 4.6. *Suppose that* v, u, w *are distinct vectors of a vector space* V *and* $v-u, u-w, w$ *are distinct vectors from* V*. Show that the set* $\{v, u, w\}$ *is a basis of the vector space* V *if and only if the set* $\{v-u, u-w, w\}$ *is a basis of* V*.*

Problem 4.7. *Find a basis of the vector space*

$$U = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \mid a+b=c,\, b-d=e,\, d+e+f=0 \right\} \subset \mathbb{R}^6. \tag{4.5}$$

Problem 4.8. *Find a subset of the set*

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{4.6}$$

that is a basis of $\mathbb{R}^3$*.*

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Problem 4.9. *Let*

$$U = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid a + b + c = 0 \right\} \subset \mathbb{R}^3. \tag{4.7}$$

Does there exist a two-dimensional subspace $W \subset \mathbb{R}^3$ such that $\mathbb{R}^3 = U \oplus W$?

Problem 4.10. *Suppose that U and W are subspaces of a vector space V, $\dim V = n$, $\dim U = k$, $\dim W = m$, $k \leq m \leq n$. What possible values can $\dim(U \cap W)$ take?*