PROBLEM SET 5

5. Linear Maps

Problem 5.1. *Which of the following maps from $\mathbb{R}^2$ to itself are linear?*

$$A\begin{pmatrix}x\\y\end{pmatrix} := \begin{pmatrix}xy\\2x\end{pmatrix}; \tag{5.1}$$

$$B\begin{pmatrix}x\\y\end{pmatrix} := \begin{pmatrix}x\\y+1\end{pmatrix}; \tag{5.2}$$

$$C\begin{pmatrix}x\\y\end{pmatrix} := \begin{pmatrix}y\\x\end{pmatrix}; \tag{5.3}$$

$$D\begin{pmatrix}x\\y\end{pmatrix} := \begin{pmatrix}x\\y^2\end{pmatrix}. \tag{5.4}$$

Problem 5.2. *Does there exist $f \in \mathrm{Hom}(\mathbb{R}^3, \mathbb{R})$ such that*

$$f\begin{pmatrix}1\\0\\0\end{pmatrix} = 1, \quad f\begin{pmatrix}0\\0\\1\end{pmatrix} = 2, \quad f\begin{pmatrix}1\\0\\-1\end{pmatrix} = 3? \tag{5.5}$$

Problem 5.3. *Is it true that a linear map is a monomorphism if and only if the image of every linearly independent subset is also linearly independent?*

Problem 5.4. *Suppose that $A \in \mathrm{Hom}(V, U)$, the subset $\{y_1, \dots, y_k\} \subset U$ is linearly independent, and there exists a set $\{x_1, \dots, x_k\} \subset V$ such that $Ax_i = y_i$, $1 \le i \le k$. Does the set $\{x_1, \dots, x_k\}$ have to be linearly independent?*

Problem 5.5. *Do there exist two different linear maps $A, B \in \mathrm{Hom}(V, U)$ such that*

$$\ker A = \ker B,\ \mathrm{im} A = \mathrm{im} B? \tag{5.6}$$

Problem 5.6. *Does there exist a linear operator $A \in \mathrm{End}(V)$ such that*

$$\ker A = \mathrm{im} A? \tag{5.7}$$

Problem 5.7. *Suppose P is a linear operator in a vector space V. Show that $P^2 = P$ if and only if the following two conditions hold:*

(1) $V = \ker P \oplus \mathrm{im} P$;
(2) $Px = x, \quad \forall x \in \mathrm{im} P$.

P is called a projection operator.

Problem 5.8. *Consider a map $A : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ defined by*

$$(A(p))(x) := p(x+1),\ \forall p \in \mathbb{R}_n[x]. \tag{5.8}$$

Is A an automorphism of the vector space $\mathbb{R}_n[x]$?

Problem 5.9. *Do there exist vector spaces V, U, and linear maps $A \in \mathrm{Hom}(V, U)$, $B \in \mathrm{Hom}(U, V)$ such that $AB = \mathrm{Id}$, but $A \notin \mathrm{Iso}(V, U)$?*

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Proposition 5.10. *Let U be a subspace of a vector space V. Define a map A from V to the quotient space V/U by*

$$Ax := x + U, \forall x \in V. \tag{5.9}$$

Show that $A \in \mathrm{Hom}(V, V/U)$; this map is called the canonical projection map.