PROBLEM SET 6

6. Matrices of Linear Maps

Problem 6.1. *Suppose that V and U are finite-dimensional vector spaces over the same field $\mathbb{F}$. Show that*

$$\mathrm{Hom}(V,U) \cong \underbrace{U \oplus U \oplus \cdots \oplus U}_{\dim V \text{ times}}. \tag{6.1}$$

Problem 6.2. *Find a vector space V, two bases α and β of V, and $A \in \mathrm{End}(V)$ such that*

$$[A]_\alpha \neq [A]_\beta. \tag{6.2}$$

Problem 6.3. *Find A^4 for the following matrices:*

(1)

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{6.3}$$

(2)

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.4}$$

(3)

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{6.5}$$

Problem 6.4. *Find the matrix of the rotation of the plane $\mathbb{R}^2$ around the origin by the angle $\frac{\pi}{3}$. You may choose any basis you prefer.*

Problem 6.5. *Denote by $\mathrm{Mat}(n \times n, \mathbb{R})$ the vector space of real-valued $n \times n$ matrices, and by U the subspace of symmetric matrices.*

a) Find a basis of U.

b) Find a subspace W such that

$$\mathrm{Mat}(n \times n, \mathbb{R}) = U \oplus W. \tag{6.6}$$

Problem 6.6. *Suppose that $\alpha = (v_1, \ldots, v_n)$ is an ordered basis of a vector space V, $\beta = (u_1, \ldots, u_m)$ is an ordered basis of a vector space U, and $A \in \mathrm{Hom}(V,U)$. Establish an equality relating A and $[A]_\alpha^\beta$ using coordinate maps $\varphi_\alpha : V \to \mathbb{F}^n$ and $\varphi_\beta : U \to \mathbb{F}^m$.*

Problem 6.7. *Consider the homogeneous system of linear equations*

$$\sum_{i=1}^{n} a_i^j x_i = 0, \quad j = 1, 2, \ldots, m. \tag{6.7}$$

Show that if $m < n$, then the system has a non-zero solution.

Problem 6.8. *Show that $A \in \mathrm{Hom}(V,U)$ is a monomorphism if and only if there exists $B \in \mathrm{Hom}(U,V)$ such that $BA = \mathrm{Id}$.*

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Problem 6.9. *Suppose that V is a finite-dimensional vector space, $A \in \mathrm{End}(V)$. Show that*

$$(A \in \mathrm{GL}(V)) \quad \Longleftrightarrow \quad (\ker A = \{0\}) \quad \Longleftrightarrow \quad (\mathrm{im} A = V). \tag{6.8}$$

Problem 6.10. *Suppose that V is a finite-dimensional vector space, $A, B \in \mathrm{End}(V)$, and $AB = \mathrm{Id}$. Does A have to be an automorphism?*

Problem 6.11. *Suppose that V and U are finite-dimensional spaces. Describe the subspaces W of the space V for which there exists $A \in \mathrm{Hom}(V, U)$ such that $\ker A = W$.*

Problem 6.12. *Suppose that V, U, W are finite-dimensional vector spaces, $A \in \mathrm{Hom}(V, U)$, $B \in \mathrm{Hom}(U, W)$. Show that the following inequality holds:*

$$\dim(\ker(BA)) \leq \dim(\ker(A)) + \dim(\ker(B)). \tag{6.9}$$

Problem 6.13. *Let V be a finite-dimensional space over a field $\mathbb{F}$. Assume that $A \in \mathrm{End}(V)$ satisfies*

$$AB = BA, \quad \forall B \in \mathrm{End}(V). \tag{6.10}$$

Does there always exist $\lambda \in \mathbb{F}$ such that $A = \lambda \mathrm{Id}$?