"Advanced Linear Algebra", Spring 2025.

Course description. Linear algebra concepts are fundamental to almost all areas of pure mathematics. This course aims to address potential gaps in understanding following a standard linear algebra course and to introduce key advanced topics, including tensor products, complexification, and Hilbert spaces. Consequently, we aim to equip students with the linear algebra knowledge necessary for advanced study and research.

We have structured the course to provide you with opportunities to prove all theorems collaboratively with your peers. Several days before each class, you will receive lecture notes containing definitions, examples, and propositions to prove. In class, we will discuss your approaches, comments, and any questions you may have. No prerequisites.

Program of the Course.

1. Groups. Fields. Vector spaces. Bases, Dimension. Subspaces, quotient spaces. ([FIS], 1.1 – 1.6, [A], ch.1, 2.)
2. Linear maps. Matrices. Multiplication of matrices. Change of basis. ([FIS], 2.1 – 2.5, [A], ch.3.)
3. Dual space. Tensor product of vector spaces, universality property. Symmetric tensors, skew-symmetric tensors. Tensor product of linear operators, its matrix. ([R], ch.14, [FIS], 2.6.)
4. Determinant. Systems of linear equations. Inverse operator. ([FIS], 3.1 – 3.4, 4.1 – 4.4, [A], ch.10.)
5. Characteristic polynomial of an operator. Cayley-Hamilton theorem. ([FIS] 5.1,5.2,5.4, [A], ch.5.)
6. Eigenvectors and eigenvalues. Jordan Canonical Form (complex case). ([FIS], 7.1 – 7.3, [A], ch.8.)
7. Complexification. Jordan form (real case). ([A], ch.9.)
8. Real inner product spaces, complex inner product spaces. ([FIS], 6.1, 6.2, [A], ch.6.)
9. Spectral theory for normal operators, unitary operators, Hermitian operators, skew-Hermitian operators, orthogonal operators, symmetric operators, skew-symmetric operators. ([FIS], 6.3 -6.6, [A], ch.7.)
10. Bilinear forms. The matrix of a bilinear form. Generalization of orthogonality notion.([R], ch.11, [FIS], 6.7)
11. Positive operators. Sylvester's Criterion. Polar decomposition. ([A], ch.7.)
12. Hilbert spaces, Hilbert bases. The Riesz Representation Theorem. ([R], ch. 13)

Recommended Textbooks.

- [FIS] S. Freidberg, A. Insel, L. Spence, Linear Algebra.
- [A] S. Axler, Linear Algebra Done Right.
- [R] S. Roman, Advanced Linear Algebra.

Grading Rules. Your final grade will be based on the following components:
Homework: 2*n points total, where 'n' is the number of graded lectures/seminars ($24 < n < 28$). *(To earn credit, submit solutions to at least two problems before each seminar and two exercises before each lecture. Each correct solution earns 1 point, with a maximum of two points awarded per lecture or seminar)*
Midterm Exam: 15 points (5 problems x 3 points each)
Final Exam: 20 points (5 problems x 4 points each)
(Both the midterm and the final exam are oral exams, open books, and open notes.)
Bonus points: Awarded for particularly elegant or complex solutions, and for correcting professor's errors (no limit)

Your final grade is determined by the ratio $(\text{your points})/(2n+35)$, according to the following scale:

A+	90%
A	85%
A-	80%
B+	75%
B	70%
B-	65%
C+	55%
C	50%
C-	45%
D	35%

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